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STABILITY IN AVIATION

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Stability in Aviation

An Introduction to Dynamical Stability as applied to the Motions of Aeroplanes

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PREFACE

Up to the present time the problem of stability has received very inadequate attention in connection with aviation. From the point of view of the practical aviator, this is, perhaps, little to be wondered at. It would scarcely be possible for him to devote months of concentrated attention to long and laborious stability investigations when it is no exaggeration to say that very frequently his success or failure depends, above all things, on the *names of the towns* at which he starts and lands. If a prize is offered for a flight from Folkestone to Flushing, it is useless for him to fly from Harwich to the Hook, even on a much more stable machine than that used by the winner of the prize.

It is hoped that the publication of this memoir will lead to aeroplane stability being made the subject of much more continuous study and investigation than has been possible in the past. A general abstract of the present investigation is contained in the introduction, which may be read with advantage before proceeding to details of a more mathematical character. The general conclusions show that there should be no difficulty in securing inherent stability, both longitudinal and lateral, in an aeroplane, by means of suitably placed auxiliary surfaces rigidly attached to the machine; but in order to achieve success the conditions of stability must be very carefully studied, and account must be taken of the effects of

the inclination of the flight path to the horizon and other causes which may affect the result seriously.

There seems a general desire on the part of many writers to minimise the dangers of instability or defective stability, and to attribute accidents to other causes. But in reading the accounts of accidents, both fatal and otherwise, that appear every few days in the daily papers, it is difficult to avoid coming to the conclusion that much of this loss of life and damage could be avoided by a systematic study of stability and certain other problems regarding the motion of aeroplanes particularised in this book.

To the mathematician who is able to devote any time to original work, a wide region of unexplored ground is opened up, of which it has, in many cases, only been possible to exhibit glimpses in these pages. There is a freshness about this region which can scarcely be found to the same extent in searches for differential equations that have not been integrated, high primes, or further additions to the large existing collection of properties of triangles and circles. I do not think any mathematician who cares to take up the subject now will experience any difficulties comparable with those which have been encountered in the preparation of the present book.

There is abundant work, too, for the student who wishes to undertake "research" for educational purposes; and "aeroplane stability" seems to me a very useful alternative for some of the branches of applied mathematics now taught in our universities.

Quite recently, much has been written regarding so-called "automatic stability," depending on the use of gyrostats, pendulums, or other movable parts. Apart from the fact that movable parts are liable to get out of order, it must be remembered that they increase the number of degrees of freedom of the machine, thus further

adding to the number of conditions which have to be satisfied for stability—a number quite large enough already. I anticipate that the successful aeroplane of the future will possess inherent, not "automatic" stability, movable parts being used only for purposes of steering.

The great difficulty in writing this book has been the newness of the subject, and the fact that the further one advances in it the greater appear the possibilities of still further exploration. The selection and arrangement of the text has thus been much more difficult than it would be in a text-book compiled from existing literature. I have tried, so far as possible, to avoid, introducing any investigation the results of which were certain to be only of academic interest; but this attempt has not made it much easier to draw the line. For example, the transformations of § 30, which are, of course, particular cases of a more general transformation, were placed on the excluded list, but their application to the pendulum experiment suggested the desirability of retaining them.

One pleasant duty remains, and this is to record my indebtedness to Mr. E. H. Harper for the valuable services he has rendered in connection with the investigations here described, and to claim priority for him in many of the results. It was Mr. Harper that directed my attention in the first instance to the important effects of the inclination of the flight path on stability, and the investigations of the lateral stability of the Antoinette type, and several other forms, are entirely due to him. The whole of the formulæ in this book have been checked by independent working by Mr. Harper and myself, and I hope that they are correct. Besides Mr. Harper, Mr. A. Ferguson and Mr. Robert Jones have revised the proofs and made a number of corrections. It is impossible to be too careful in a

matter where a mere slip of a sign might change stability into instability, and for this reason it is desirable that readers should, as far as possible, keep a further check on the formulæ, even in spite of these precautions.

Last but not least my thanks are due to Prof. R. A. Gregory not only for his unfailing kindly advice and assistance in the preparation of the text, but also for his offer to include a volume on the investigation of stability in the series of *Science Monographs*, of which he is the Editor. There are probably many instances, of which this is one, in which a volume of this kind presents a more appropriate medium for the publication of results of investigation than the proceedings or transactions of a scientific society. Had it not been for Prof. Gregory's timely invitation, the theory of stability would have been developed much less fully than has now been done in these pages.

G. H. BRYAN.

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LIST OF ILLUSTRATIONS OF AEROPLANES

*** The following illustrations, distinguished by Roman numerals, will, it is hoped, enable the reader to visualise the systems of planes described in the text, or systems resembling them, better than would be possible from the diagrams bearing Arabic numbers. They also suggest material for further research. Under each figure will be found a brief statement of the particular points which it illustrates in connection with stability.

STABILITY IN AVIATION

CHAPTER I

INTRODUCTION AND SUMMARY

1. The development of the aeroplane has opened up a large number of interesting problems in both theoretical and experimental mechanics.

One of these problems is the investigation of the pressures on plane and other surfaces moving in a given manner through a fluid medium such as air. This problem had attracted the attention of mathematicians long before applications to aërial navigation had reached the stage of practicability. The earliest attempt at such an investigation was that due to Newton, who assumed that the resultant pressure on a lamina could be measured by the momentum or change of momentum imparted (per unit time) by the lamina to the particles of air with which the lamina came into contact. This led to the so-called "sine squared" law, according to which the pressure on a plane lamina appeared to be proportional to the square of the *normal* velocity of the lamina relative to the surrounding medium. On the Kinetic Theory of Gases Newton's assumption might reasonably be expected to hold good in the case of a rarefied gas, in which the mean free path of the molecules was large compared with the breadth of the lamina. The second method employed was the theory of Discontinuous Fluid Motion, introduced by Helmholtz

and Kirchhoff, which forms the subject of a memoir by Sir George Greenhill, published by the Advisory Committee for Aeronautics.[1]

Divergences between theory and observation necessarily arise from the character of the assumptions required to bring the problem within the reach of analytical methods. The failure of Newton's Law showed that it was impossible to estimate the pressure without taking into account the currents set up in the medium in the neighbourhood of the lamina. In spite of this, many writers, even at the present time, seek to establish formulæ for the pressure on moving planes by reasoning which is more or less Newtonian in its fundamental characteristics, but it will be evident that such methods should only be adopted with great caution, the results being treated more or less in the light of empirical formulæ which must stand or fall on experimental evidence.

In the case of the theory of discontinuous motion, we are limited by mathematical difficulties to the consideration of a perfect incompressible fluid; and divergences between theory and experiment necessarily arise from the differences between the physical properties of such an assumed medium and those of air, or of water as the case may be. Moreover, the soluble problems in general only refer to two-dimensional motions, so that in any case the necessary conditions would only be satisfied approximately in the case of aeroplanes the so-called "aspect ratio" of which is small, and the theory would fail in the neighbourhood of the extremities of such planes.

In this particular problem a suitable balance between theory and experiment has been sought by the publication

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[1] Report on the Theory of a Stream Line past a Plane Barrier, and of the Discontinuity arising at the Edge, with the Application of the Theory to an Aeroplane. By Sir George Greenhill, F.R.S. Pp. 96+106 figs. (Advisory Committee for Aeronautics Reports and Memoranda, No. 19.) (London: H.M. Stationery Office: Wyman and Sons, Ltd.). Price 5s.

of Sir George Greenhill's report by the Advisory Committee which has been organising the construction of apparatus for the experimental determination of air pressures on moving planes.

2. In the problem of aeroplane stability here discussed, we are concerned with the motions or changes of motion set up in the flying machine itself by these pressures and by the other forces acting on it when equilibrium in steady motion is disturbed. This problem, unlike the previous one, is a direct application of the principles of theoretical mechanics involving the use of the equations of Rigid Dynamics. About the validity of these, no question can at present arise. The only further assumptions are the expressions for the air pressures on the planes and other parts of the machine; and the closeness of agreement between theory and observation in any particular calculation will depend on the degree of approximation to which these pressures are represented by the formulæ assumed for them.

The present subject has hitherto not received so much attention as it deserves, one reason, probably, being the complexity of the algebraic formulæ which present themselves at the outset, and another, doubtless, the success with which aviation has been accomplished on machines the stability of which has not been studied. When the Wright Brothers made their first flights, the question at issue was one of motive power and weight, and an important condition for success was the reduction of weight and head resistance by avoiding the introduction of any more auxiliary surfaces than were absolutely necessary. If this involved instability its effects could be overcome by skill and experience, and in particular lateral control was obtained by warping the wings. Since then many attempts have been made to circumvent the Wright patents, with the result of diverting attention from the question of inherent lateral stability.

The Bryan-Williams paper of 1903[1] was, I believe, the first attempt to direct attention to the mathematical aspect of the question, and it was intended to be suggestive rather than conclusive. The late Captain Ferber, whose papers were published soon afterwards, investigated independently the equations of motion of an aeroplane, and he subsequently applied the method of small oscillations to the problems of both longitudinal and lateral stability. This constituted the first attempt at a theory of lateral stability; unfortunately, however, this last is defective in several respects, and the results probably fall very wide of the mark when applied to aeroplanes the main surfaces of which are of considerable span, as is usually the case.

In Mr. Lanchester's second volume on Aerodonetics, which appeared in 1908, the problem of stability, both longitudinal and lateral, is treated from a highly original point of view, the use of the equations of Rigid Dynamics being practically avoided.

Of more recent contributions we have the papers of Prof. Marcel Brillouin and Dr. H. Reissner, the latter of whom has considered the problems of both lateral steering and lateral stability; while the Government Blue Book (Cd. 5282) contains abstracts of papers by Lieut. Crocco on the steering and stability of dirigibles, and by M. Soreau on similar problems for an aeroplane, with especial reference to longitudinal stability. This group of papers arrived after the investigations here discussed had been completed, and only a brief comparison of the results has been attempted near the end of the present work.

3. In this book the six equations of motion are first written down, and then applied to the small oscillations about steady motion of a body, such as an aeroplane, moving in a resisting medium.

I first intended to measure forces in dynamical units, and perhaps the equations would have been a little simpler

[1] *Proc. R. S.*, June, 1903.

if these had been retained, but for the sake of convenience, gravitation units of force have been substituted, the equations being based upon the "homogeneous" system,

$$\text{Force} = \text{Weight} \times \frac{\text{acceleration}}{g},$$

which is now recommended by teachers of elementary mechanics, and avoids both the poundal and the slug. The advantage arises when air pressures are measured in pounds or grammes weight per unit of area instead of poundals or dynes.

In the case of an aeroplane which is symmetrical with respect to a plane which is vertical in steady motion, the small oscillations fall into two groups each determined by three equations of motion, and no further separation is at the outset possible.

The first group represents motions in the vertical plane on which longitudinal stability depends. They may be better described as "symmetrical" oscillations.

The second group determines sideways rotations and displacements to one side or the other of the plane of symmetry. These should be called "asymmetric oscillations." On the nature of these depends the lateral stability of the machine. Some writers attempt to separate the three motions here considered, and to treat separately the stability for rolling motions and "directional" stability, a mistake which Lanchester avoids by using the term "rotative stability." It is important that the interdependence, not only of the two rotational oscillations, but also of sideways displacements, should be taken into account. It is only through the latter displacements that gravity can have any effect on the direction of an aeroplane. Without such an effect there can be no tendency on the part of an aeroplane to right itself after it has heeled over sideways, and in the absence of such a tendency the aeroplane, if it has a tilt to one side, will be

liable to "side slip" with uniform acceleration. In most existing aeroplanes this tendency is counteracted by warping, but it does not follow that because this plan can be successfully used it is necessarily the only or the best plan.

For the second kind of stability the best name would be "asymmetric" stability. We shall sometimes, for shortness, use the term lateral stability to denote the stability dependent on this group of displacements in spite of the term having been used in a more restricted sense by other writers.

4. The effects of the air resistances in each group of equations are represented by nine coefficients (or eighteen in all) which may be called the "resistance derivatives." On elimination, the frequencies and logarithmic decrements (or in the case of instability, logarithmic *increments*) of the *symmetrical* oscillations are determined by an equation of the fourth degree in λ where the displacements are assumed proportional to $e^{\lambda t}$, this equation involving the "symmetrical" resistance derivatives. The conditions of stability require that the coefficients of the biquadratic, as well as Routh's discriminant, should be positive.

Exactly similar conditions apply to *asymmetric* oscillations, which again lead to a biquadratic in λ involving the nine "asymmetric" resistance derivatives and corresponding conditions of lateral stability.

The actual co-ordinates of the centre of gravity, as well as the azimuth, do not occur in the equations of motion, though the corresponding velocities do so. The reason is that the conditions of equilibrium of an aeroplane, like those of a ship, do not depend on its position or compass bearing. For variations of these, equilibrium is neutral.

The resistance derivatives are proportional to the velocity of the aeroplane when in steady motion.

The problem as now before us resolves itself into (1) the determination of the expressions for the resistance deriva-

tives of the planes based on the available formulæ or data regarding the pressures on them; (2) the evaluation and simplification of the coefficients in the biquadratics and their discriminant; (3) analysis of the general character of the oscillations; (4) separate discussion of the effects of various subsidiary causes which may affect the stability of an actual flying machine.

The first object to be kept in view is to evolve something like order out of the chaos of algebraical symbols that present themselves at the outset of the inquiry. This can best be done by a method of successive approximation and building up, the problem being at first treated subject to the simplest possible assumptions, but the methods adopted being sufficiently general and elastic to allow of subsequent modifications, corrections, and extensions of the most general character. In this book attention is concentrated on the mathematical aspect of the problem for several reasons. In the first place, there is no obvious alternative between developing the mathematical theory fairly thoroughly and leaving it altogether alone; any attempt at a *via media* would probably lead to erroneous conclusions. In the second place, the formulæ arrived at, even in the simplest cases, are such that it is difficult to see how they could be established without a mathematical theory. In the third place, there is probably no lack of competent workers interested in the practical and experimental side of aviation, and under these conditions it is evident that the balance between theory and practice can be improved by throwing as much weight as possible on the mathematical side of the scale. Lastly, it is hoped to advocate the claims of aeroplane equilibrium and stability as an educational subject suitable for study in our Universities alongside with such branches of applied mathematics and mathematical physics as hydrodynamics, and particle and rigid dynamics.

5. Starting with longitudinal or symmetric stability, the

available experimental data refer to the air pressures and positions of the centres of pressure of plane and other areas moving through air with given relative velocities of *translation*. If a lamina has in addition a motion of *rotation*, as when an aeroplane begins to pitch, the effects of this rotation appear to be at present unknown. In all existing theories of stability they have been neglected, and at the time of writing no indications have reached us of the matter receiving attention at the Government Laboratory. The effects referred to introduce two constants or coefficients into the equations, and these I call *rotary derivatives*.

Apart from mathematical considerations, determination of these effects would be a matter of common prudence, as rotations may easily put an aviator into an awkward predicament, especially when near landing. The experimental problem may if desired be reduced to the determination of two points which are here shown to exist in the plane of the lamina, such that rotation about one does not affect the resultant pressure, while rotation about the other does not displace the centre of pressure. Two methods of experimenting are suggested.

The difficulty is meanwhile obviated by the use of narrow planes for which the rotary derivatives are negligible, stability being secured by auxiliary surfaces, such as a tail. But in this case the variations in the centre of pressure will also, in general, be small. We are thus led to the **theory of narrow planes flying at small angles.** This theory forms the best starting-point for the systematic study of aeroplane motions, in just the same way that the definition of a perfect fluid forms the starting-point in hydrodynamics, the theory of refraction of narrow pencils at small angles of incidence in geometrical optics, or the theory of elliptic motion in planetary theory. According to this hypothesis, not only are rotatory derivatives and variations of the centre of pressure neglected, but the

pressure is assumed proportional to the sine of the angle of attack, or what is the same thing, to the product of the resultant and the normal velocity.

A conclusion described as the Principle of Independence of Height shows that, subject to certain limitations, the stability of an aeroplane is but very slightly affected by raising or lowering its planes; for this reason there is practically nothing to choose between monoplanes and biplanes in this respect.

6. The section on "Graphic Statics of Longitudinal Equilibrium" stands by itself apart from the rest of the book, and will, it is hoped, be found instructive. It is sometimes a little difficult to see how the conditions of equilibrium of a flying machine are altered by varying the propeller thrust or altering the inclinations of the planes, and the object is to exhibit the results by means of geometrical constructions. This section might very well be taught to university students.

7. The simplest case of longitudinal stability is that of a single lifting plane with a "neutral" auxiliary tail propelled horizontally by a constant horizontal thrust passing through the centre of gravity. The condition of stability agrees with Lanchester's result. By the use of the equations of equilibrium this condition may be put into several alternative forms, one of these being independent of the velocity of propulsion. By the further use of approximate methods the roots of the biquadratic are separated, the short and long oscillations discussed, and their general characteristics described. It is shown that Lanchester's method only applies to the long oscillations, but, fortunately for Lanchester, these are the oscillations on which stability mainly depends.

In a non-horizontal path, the stability varies very greatly with the inclination of the flight path to the horizon. The important part played by this inclination was found out, both for longitudinal and for lateral stability, by

Mr. E. H. Harper. A glider has greater longitudinal stability (*cæteris paribus*) than a horizontally propelled aeroplane; and when an aeroplane rises the longitudinal stability rapidly falls off. Under simple assumed conditions, longitudinal instability sets in when the tangent of the elevation has some value *less* than twice that of the angle of attack.

I find, however, that longitudinal stability is in general increased by head resistance, and a further increase occurs when the propeller thrust is a function of the velocity which decreases with an increase of velocity. Both these effects are conveniently represented as being equivalent, in their effects on stability, to a depression of the direction of the flight path, through an angle for which a formula is given. The machine can therefore rise at a steeper angle than before without losing stability.

By a *doubly lifting system*, I mean two lifting planes placed tandem, or, what comes to the same thing, a front lifting plane and a lifting instead of neutral tail. (In stability investigations it must be remembered that two superposed planes practically count as *one* of the same total area.) Here a property is proved, which may be called the *principle of equivalent systems*, which enables the combination of two planes in question to be replaced by an "equivalent" single lifting plane with neutral tail, the two systems having the same conditions of equilibrium and also of stability and, in addition, the same total combined area of their planes.

Without this substitution the stability of doubly lifting aeroplanes involves the simplification of very unwieldy formulæ. For the mathematical student the properties of equivalent systems and their invariants afford some interesting applications of determinants. From a technical point of view, the conclusion to be drawn is that there is no advantage or disadvantage in a lifting tail as compared with a neutral one except that the neutral tail will

have to be the longer of the two. In an actual aeroplane where the pressure on the rear planes is affected by the wash of the front ones, the machine with the neutral tail would, probably, have the greater lifting capacity.

In the following sections, the necessary modifications are considered for propeller axis not passing through centre of gravity, variations in the position of the centre of pressure as the angle of attack varies, the principal corrections for cambered planes, effects of tangential or frictional resistances, raised planes, deviations from the sine law of resistance, "wash" on tail plane produced by front one, and rotary derivatives. The discussion of these corrections might be extended almost indefinitely, and, in order to draw the line somewhere, I have in some cases thought it sufficient to indicate the methods only.

The effect of variations in the position of the centre of pressure calls for a little comment in view of the discussions that have already taken place on this subject. In the sections dealing with this effect two cases have been considered; one is the case where stability is mainly dependent on the action of an auxiliary tail plane (which in this case must not be parallel to the front plane) and the effect of the variations in the centre of pressure takes the form of a small correction in the result. The second case refers to the stability of a single plane, which is entirely dependent on the variations in question. (Of course the latter solution is not altogether satisfactory, owing to the fact that the unknown rotary derivatives have been neglected.) Now, when the conditions of stability are taken in conjunction with the conditions of *equilibrium*, it is shown that the stability depending on the tail action is independent of the velocity of the aeroplane, but the stability of the tailless aeroplane decreases as the velocity increases. This confirms the view that the use of a tail affords the more efficient means of securing inherent longitudinal stability.

As has been observed, the conditions of longitudinal stability can be exhibited in various forms by making use of the equations of equilibrium, so that two apparently widely different conditions may in this way be equivalent.

A "three plane system," such as a lifting plane with auxiliary surfaces fore and aft, is shown, in certain circumstances, to possess advantages over a two plane system in respect of stability, especially when rising in the air.

It will thus be seen that the mathematical method is not limited in its application to the simplest case of an ideal aeroplane, but that account can be taken of practically all the modifications occurring in an actual flying machine as soon as the necessary data become known. And in any case we find out which modifications increase, and which diminish, the stability. At the outset of the investigation a certain determinant was found to vanish, and this fact was of great importance in simplifying the analysis which, but for this, appeared of great difficulty. In the discussion of the "three plane system," the fact that this determinant does not *now* vanish is important in indicating the greater stability of such a combination.

8. Lateral or Asymmetric Stability presents a difficult problem possessing many interesting features. In this case the couples set up by rotations of the main planes depend on the *span* instead of the breadth of the planes, and instead of being negligible play a very important part in the stability. To obtain expressions for them which may at least be regarded as approximate, a further assumption has to be made regarding the distribution of pressure on them, this assumption being regarded as a further property which is to be implied in the definition of "narrow planes gliding at small angles." The couples in question are the "turning moments" which are so well known in aviation, and are usually counteracted

by operating on "ailerons" or warping. In lateral stability, however, we have to consider separately in turn the effects of "straight" main planes, auxiliary vertical surfaces or rudders which may be called "fins," "bent up" wings, and slanting up planes fixed at the extremities of the main planes which may be called "stabilizers."

An aeroplane having straight (*i.e.* unbent up) planes (without fins or stabilizers) is devoid of lateral stability. Moreover, a horizontal auxiliary tail plane, such as is used for maintaining longitudinal stability, does not enter into the equations of lateral motion unless its span be considerable.

If a single vertical fin is added, there is great difficulty in reconciling two of the stability conditions. If the fin is placed in front and near the level of the centre of gravity one condition is satisfied but not the other; if placed behind and near the same level, the second condition is satisfied but not the first. Stability might possibly be got by placing the fin above the centre of gravity and slightly in front, the height above being large compared with the distance in front, but a small change in the motion of the aeroplane would be liable to render it unstable. The objections to a fin raised considerably above the C.G. are evident. That most existing aeroplanes are laterally unstable is a necessary further deduction.

This difficulty can be got over by the use of two fins instead of one. We have here to investigate the difference between the effect of several fins and that of a single fin of the same total area placed at their centre of pressure, and as the expressions involved are the moments of inertia of the areas of the fins, we transform these by the "theory of parallel axes." We find that the horizontal distance between the fins has much more effect on stability than a difference in height.

Probably the best arrangement is one in which the aeroplane is provided with two fins placed fore and aft, and

raised somewhat above the centre of gravity, so that their centre of pressure is vertically above the centre of gravity, or nearly so. This arrangement is shown to possess a wide range of stability which is practically unaffected by the angle which the direction of flight makes with the horizon. The condition for stability requires that the height of the fins should not exceed a certain limit.

If instead we place the fins at the same level as the centre of gravity, their centre of pressure must be in front of it, and the stability is secured by making the distance between the fins large and the distance of the centre of pressure in front of the centre of gravity small. This arrangement will be stable in horizontal or ascending flight, as also in a gentle descent, but the aeroplane will become laterally unstable and liable to turn round sideways if it dives downwards at an angle to the horizon equal to or greater than twice the angle of attack.

A third arrangement, which, like the last, was investigated in the first place by Mr. Harper, has a vertical tail fin and a second vertical fin directly above the centre of gravity. For stability the tail must not be less than a certain length, and in this case it is found that stability *decreases* when the aeroplane is *ascending*, thus leading to a limit to the angle of *elevation*.

The general character of the lateral oscillations is discussed, also the general effects of couples due to head-resistance, twin screws, friction, and camber.

We next consider the effects of "stabilizers" (or bent up planes attached to the tips of the wings, or, in fact, wings with bent up tips), and bent up wings in general. It is natural to suppose that a pair of stabilizers is equivalent to a vertical fin; this is, however, shown to be only true in certain conditions, such as when they are placed at grazing incidence with the line of flight, or are neither before nor behind the centre of gravity. The two solutions which follow are due to Mr. Harper. The first

discusses the stability of an aeroplane fitted with stabilizers, the second the "Antoinette" type of aeroplane, furnished with bent up wings and a tail-fin. Mr. Harper has shown that lateral stability in this case can be obtained in two ways. Neglecting the tail, it is sufficient to raise the dihedral angle of the wings *above* the centre of gravity, but in this case there is a superior limit to its height. Or again without raising the wings the tail must be made not less than a certain length. In the first method stability falls off when the aeroplane is descending; in the second it falls off when rising, so that by a judicious combination of raised wings and tail, stability may be made practically independent of the elevation of the flight path.

9. Under the heading of "General Conclusions," no attempt has been made to give a detailed summary of the work, this having been done in the present introduction. The discussion refers chiefly to the diversity of opinion which now exists as to the advantages and disadvantages of inherent stability.

There is, however, an element of indeterminateness in the arguments relating to this particular point, arising from the fact that a complete investigation of aeroplane motions involves the solution of other problems besides that of stability. A list of some of these problems is given at the end. While some of them afford materials for experimental research, the list should afford the most sceptical reader evidence as to the necessity of a large amount of further original work of a purely mathematical character, in which all that is required is the deduction of conclusions from definite stated hypotheses. At present there is widespread belief that the methods of exact science cannot be brought to bear on the study of aeroplane motions owing to the uncertain conditions to which they are subjected. But after all the cause of the uncertainties probably lies in the fact that the study of aeroplane motions has not received so much attention on the part of

mathematicians as has been given to other problems of a similar character.

10. A brief discussion is given, dealing with investigations of an allied character by other writers. It will be seen that in several cases erroneous conclusions have been arrived at through not starting with sufficiently general hypotheses, a frequent cause of error being failure to recognise the interdependence of the three components of lateral motion. In this respect Lanchester's work forms a conspicuous exception, although he has approached the subject from an entirely different direction from that here adopted, and has avoided the use of the actual equations of motion.

11. If we consider only the problem of stability, in which it has been attempted to arrive at some approach to a definite understanding in this book, it will be seen that this problem presents many complexities which it would have been very difficult to foresee from experimental evidence alone. Indeed, I should be inclined to think that the difficulty was almost as great as that which would be experienced by anyone who endeavoured to explain the motions of the solar system without the assistance of the theory of gravitation and the fundamental equations of dynamics. The dependence of the stability of aeroplanes on the inclination of their flight-paths to the horizon is very difficult to explain from first principles, but an attempt at such an explanation has been given in a note.

These difficulties and the large number of problems still awaiting solution must be my justification for repeating the plea for the recognition of motions of aeroplanes as a subject for study, research, and original work by teachers and students of mathematical and physical science in our Universities.

As a branch of higher applied mathematics the study of aeroplane motions has been sadly neglected. The vaguest notions still prevail even as to the very meaning of

stability. The subject presents claims of the most urgent and pressing character for systematic study and investigation on similar lines to those which have been followed in the development of other branches of mechanics and mathematical physics, and it is only by the co-operation of University teachers in mathematics and physics that the present anomalous conditions can be remedied. No less strong are the claims of this study from an educational point of view. It will be seen that the subject matter contained in these pages contains several interesting methods, including applications of properties of determinants, the use of approximations, and the "principle of equivalent systems" as applied to double-lifting aeroplanes; further, the introduction, step by step, of corrections which need in no respect stop short of practical requirements, and may be possibly carried to a stage even beyond what is sufficient for ordinary needs.

To see how matters stand, let us compare the study of aeroplane motions with some subject which is already largely studied and taught in our Universities, say hydrodynamics. We have here a branch of applied mathematics which has been developed on the theoretical side quite independently of the requirements of the hydraulic engineer. Many of its problems refer to hypothetical motions in an indefinitely extended ideal medium, and are far removed from any possibility of practical application, yet no one questions the educational value of the training afforded by their study, even to engineering students.

Now any honours student who devotes a year or more to the systematic study of hydrodynamics will, at the end of that time, have little prospect of finding any problem, practical or unpractical, which has not already been solved, and which he can approach with any reasonable chance of obtaining a solution or result worth publishing. The present subject, on the other hand, literally bristles with unsolved problems, and the difficulty is not so much that

the problems are insoluble as that their solutions are long and tedious. For this reason it is important that the number of persons interested in the subject should no longer be *nearly countable on the fingers of one hand.*

12. There still remains the question of "*Examples.*"

An almost unlimited number of these can be obtained by applying the formulæ and methods of this book to existing aeroplanes the dimensions of which will be found in the principal aeronautical journals and manuals. Possibly readers of a practical turn of mind might have preferred to see some of the calculations worked out here. This, however, is quite unnecessary as the calculations are easy and well within the powers of a fair student of B.Sc. standard. In the case of numerical calculations there is no need to limit the work to substituting values in the conditions of stability. It is easy to obtain the coefficients in the biquadratic and to determine approximate values for the roots. In this way the periods and logarithmic decrements of the oscillations can be found, or in cases of instability their logarithmic *increments.* Thus, even if it is decided to sacrifice some of the conditions of stability in the construction of an aeroplane, the effects of the sacrifice can be made the subject of calculation. As different new forms of aeroplane are constructed fresh examples are continually suggesting themselves. For this reason it would have been futile to take up any of these pages with calculations which would soon be out of date. Examples on longitudinal stability were worked out by Mr. Williams in our joint paper of 1903 referred to above.

CHAPTER II.

FUNDAMENTAL PRINCIPLES.

General Equations of Motion.

13. In investigating the motion of aeroplanes we start, of necessity, by writing down the general equations of rigid dynamics.

We take as origin the centre of mass of the aeroplane, and choose three axes mutually at right angles, fixed relatively to the aeroplane and moving with it in space. We use the following notation :

W, weight of aeroplane.
A, B, C, moments of inertia about the axes.
D, E, F, corresponding products of inertia.
u, v, w, components of translational velocity.
p, q, r, components of angular velocity.
h_1, h_2, h_3, components of angular momentum.

Then we have the following equations of motion (using homogeneous gravitation units) :

$$W\left(\frac{du}{gdt}+\frac{qw}{g}-\frac{rv}{g}\right)=\text{Accelerating force along axis of } x \quad . \quad (1)$$

and two similar, also

$$\frac{dh_1}{gdt}+\frac{qh_3}{g}-\frac{rh_2}{g}=\text{Accelerating torque about axis of } x \ . \ (1a)$$

while the expressions for the angular momenta are in the most general case :

$$\begin{aligned} h_1 &= Ap - Fq - Er \\ h_2 &= Bq - Dr - Fp \\ h_3 &= Cr - Ep - Dq \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

14. Suppose, in the first place, that the aeroplane is flying steadily in a horizontal straight line. We take this line as our axis of x, and shall call it the *line of flight* (the centre of gravity being origin as postulated), a line drawn vertically *downwards* as axis of y, and a horizontal line perpendicular to these as axis of z. In the case of a symmetrical aeroplane, such as commonly exists in practice, the plane of x, y will be the *plane of symmetry*, so that $D=0$ and $E=0$. It is here assumed that gyrostatic effects due to the rotatory inertia of the propeller are neglected. If it be necessary to take account of them this must be done in a subsequent investigation, by insert-

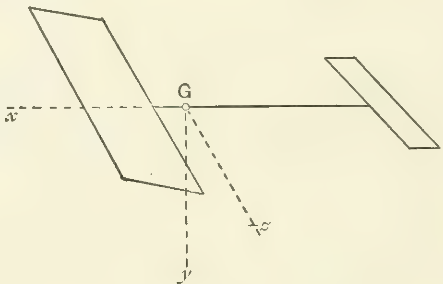

Fig. 1.

ing additional terms in the equations of motion. For an aeroplane with two propellers rotating in opposite directions the gyrostatic effects annul each other.

If the aeroplane be turned into any other direction this latter direction could be specified by Euler's angular coordinates, but as these are not well suited for the study of small oscillations, the following scheme is preferable.[1]

Starting from an initial position, imagine the aeroplane first rotated about the axis of y, through an angle ψ (this

[1] In the system as specified in Routh's "Rigid Dynamics" and elsewhere, the axes are first rotated about the axis of z, then about the axis of y, then again about the axis of z. The objection to this specification is that if the system receives a small rotation about the axis of x, this cannot be represented by *small* values of the angular co-ordinates.

merely changes its orientation), then about the axis of z through an angle θ, lastly about the axis of x through an angle ϕ. The cosines of the angles between the old axes

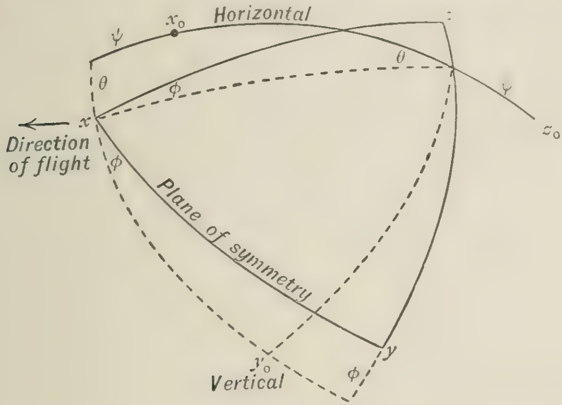

FIG. 2.

x_0, y_0, z_0, and the new axes x_1, y_1, z_1, are given by the following scheme :—

	x_1	y_1	z_1
x_0	$\cos\theta\cos\psi$	$\sin\phi\sin\psi - \cos\phi\cos\psi\sin\theta$	$\cos\phi\sin\psi + \sin\phi\cos\psi\sin\theta$
y_0	$\sin\theta$	$\cos\theta\cos\phi$	$-\cos\theta\sin\phi$
z_0	$-\cos\theta\sin\psi$	$\sin\phi\cos\psi + \cos\phi\sin\psi\sin\theta$	$\cos\phi\cos\psi - \sin\phi\sin\psi\sin\theta$

(3)

If we assume, however, that the old axis of x is in the same vertical plane as the new one, so that the change of orientation ψ is zero, the scheme assumes the simpler form,

	x_1	y_1	z_1
x_0	$\cos\theta$	$-\sin\theta\cos\phi$	$\sin\theta\sin\phi$
y_0	$\sin\theta$	$\cos\theta\cos\phi$	$-\cos\theta\sin\phi$
z_0	0	$\sin\phi$	$\cos\phi$

(3a)

The angular velocities, p, q, r, are given in terms of $\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$, by the scheme,

$$
\begin{aligned}
p &= \dot{\phi} + \dot{\psi} \sin \theta \\
q &= \dot{\theta} \sin \phi + \dot{\psi} \cos \theta \cos \phi \\
r &= \dot{\theta} \cos \phi - \dot{\psi} \cos \theta \sin \phi \quad . \quad . \quad . \quad (4)
\end{aligned}
$$

reducing, as they should, in the case of $\theta = 0$, $\phi = 0$ to $p = \dot{\phi}$, $q = \dot{\psi}$, $r = \dot{\theta}$.

It will be observed that θ *is the inclination of the line of flight to the horizon, taken positive when the aeroplane*

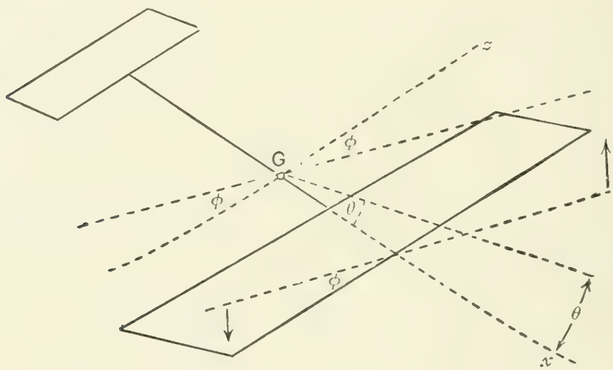

FIG. 3.

is flying downwards, and ϕ *is the angle through which the plane of symmetry is turned out of the vertical position when the aeroplane turns over sideways.*

15. The **impressed forces and couples** are due to (i) gravity, (ii) propeller thrust, (iii) air resistance.

The components of gravity along the axes are

$$W \sin \theta, \qquad W \cos \theta \cos \phi, \qquad -W \cos \theta \sin \phi \quad . \quad . \quad (5)$$

the corresponding moments all vanishing.

For the components of propeller thrust, we notice that in steady horizontal flight this acts usually in the direction of motion, and we are therefore justified in choosing its direction as our axis of x. (The more general case is

discussed later). Let the thrust be H, and let it act at a perpendicular distance, h, below the origin, then its components are specified as below:

Point of application	0	h	0
Force	H	0	0
Torque	0	0	$-Hh$

(6)

For the components of air resistance we assume that these reduce to forces X, Y, Z, and couples L, M, N, *and that these are taken positive when they tend to retard the corresponding motions of translation and rotation.* Thus, *e.g.*, the force acting along the the *positive* direction of the axis of x instead of being represented by X as is usual in text-books, is represented by $-X$. This convention is made because the forces and couples are of the nature of resistances, and it really appears to work out more satisfactorily than the ordinary convention.

We now write the equations of motion:

$$W\left(\frac{du}{gdt} + \frac{qw}{g} - \frac{rv}{g}\right) = W\sin\theta + H \quad - X \qquad (7u)$$

$$W\left(\frac{dv}{gdt} + \frac{ru}{g} - \frac{pw}{g}\right) = W\cos\theta\cos\phi \quad - Y \qquad (7v)$$

$$W\left(\frac{dw}{gdt} + \frac{pv}{g} - \frac{qu}{g}\right) = -W\cos\theta\sin\phi - Z \qquad (7w)$$

$$A\frac{dp}{gdt} - F\frac{dq}{gdt} + (C - B)\frac{rq}{g} + F\frac{pr}{g} = \quad - L \qquad (7p)$$

$$B\frac{dq}{gdt} - F\frac{dp}{gdt} + (A - C)\frac{pr}{g} - F\frac{qr}{g} = \quad - M \qquad (7q)$$

$$C\frac{dr}{gdt} \qquad + (B - A)\frac{pq}{g} - F\left(\frac{p^2 - q^2}{g}\right) = - Hh - N \quad (7r)$$

If the axis of x is a principal axis of inertia, $F = 0$ and the equations are greatly simplified.

Steady motion and small oscillations.

16. Let the aeroplane be descending with uniform velocity U in the direction of the axis of x, and let this axis make a constant angle, θ_o, with the horizon. It will follow from conditions of symmetry that the plane of x, y is vertical, and therefore $\phi = 0$. In this case u is constant

and equal to U, whereas v, w, p, q, r, the other velocity components, are all zero.

Let the components of resistance and thrust in this case be denoted by the suffix o, thus X_o, Y_o, Z_o, L_o, M_o, N_o, and H_o. The equations for steady motion become

$$\begin{aligned}
0 &= W \sin\theta_o + H_o - X_o \\
0 &= W \cos\theta_o - Y_o \\
0 &= -Z_o \\
0 &= -L_o \\
0 &= -M_o \\
0 &= -H_o h - N_o \quad . \quad . \quad . \quad . \quad (8)
\end{aligned}$$

These equations merely state the fact that the forces are in equilibrium among themselves. The conditions of equilibrium may be studied by any methods that are most convenient, for example, by the use of graphical methods.

If the thrust, H, instead of being parallel to the axis of x is inclined to it at an angle η, the first two equations are replaced by

$$0 = W \sin\theta_o + H_o \cos\eta - X_o \text{ and } 0 = W \cos\theta_o + H_o \sin\eta - Y_o$$

If now the aeroplane begins to perform *small* oscillations in still air, we assume that in this case the velocity along the axis of x changes from U to $U+u$, where u is small; further, that the other velocity components, v, w, p, q, r, are all small. In the "theory of small oscillations" of dynamics we suppose that squares and products of u, v, w, p, q, r are negligible.

The resistances, X, Y, Z, L, M, N, are functions of the velocity components, $U+u$, v, w, p, q, r, and the further assumption in dealing with small oscillations is that to a first approximation these resistances are expressible in the form

$$X = X_o + uX_u + vX_v + wX_w + pX_p + qX_q + rX_r \quad . \quad (9)$$

i.e., that they are linear functions of the small quantities u, v, w, p, q, r (the coefficients being denoted by X_u, X_v . . . Y_u, Y_v . . . etc.). This assumption, being commonly made in treatises on theoretical mechanics, as a first approxima-

tion where small oscillations are concerned, need not be here discussed.

If the aeroplane is moving in a turbulent atmosphere, there will be additional forces and couples caused by the gusts of wind; in this case, we must add to the expressions, terms X_1, Y_1, Z_1, L_1, M_1, N_1, varying with the time, representing these disturbing forces and couples. It is clear that the first step in the discussion is to investigate what happens to an aeroplane when left to itself, *i.e.*, when these components are all zero, and the aeroplane is said to be performing *free* oscillations.

Separation of the two groups of oscillations.

17. The six components X, Y, Z, L, M, N, being *each* expressed as a linear function of u, v, w, p, q, r (for purposes of approximation), there would be altogether thirty-six coefficients, such as X_u, X_v, Y_u, if we knew nothing about the symmetry of our aeroplane. But if this is symmetrical, the coefficients reduce to eighteen, the other eighteen vanishing. Thus X, Y, N do not occur with suffixes w, p, q, and on the other hand Z, L, M do not occur with suffixes u, v, r. It is easy to verify these statements, as the following instances will show:—

If, for example, X_w were different from zero (say positive) then a sideways velocity, w, from right to left (looking in the direction of flight) would cause an increase, wX_w, in the head resistance X, whereas a sideways velocity in the opposite direction would cause a decrease.

As another instance, if Z_r were different from zero and positive, then if the machine were to dip down with angular velocity r, it would experience a sideways resistance, rZ_r, which would be reversed in direction if it were to tip up, the angular velocity, r, being reversed. The reader may be left to verify the other cases which present themselves.

In the small oscillations, moreover, θ will differ from θ_o by a small quantity ϵ, and ϕ will be small, so that we shall write:

$$\begin{aligned} \sin\theta &= \sin\theta_o + \epsilon\cos\theta_o, \qquad & \cos\theta &= \cos\theta_o - \epsilon\sin\theta_o \\ \sin\phi &= \phi & \cos\phi &= 1 \end{aligned} \qquad (10)$$

Finally we suppose the new propeller thrust to be $H_o + \delta H$, h remaining constant. When all these substitutions are made, $U+u$ being of course written for u, the equations (7) give

$$\begin{aligned}
W\frac{du}{gdt} &= W(\sin\theta_o + \epsilon\cos\theta_o) + H_o + \delta H \\
&\quad - X_o - uX_u - vX_v - rX_r \quad . \quad . \quad . \qquad (11u) \\
W\left(\frac{dv}{gdt} + \frac{rU}{g}\right) &= W(\cos\theta_o - \epsilon\sin\theta_o) \\
&\quad - Y_o - uY_u - vY_v - rY_r \quad . \quad . \quad . \qquad (11v) \\
W\left(\frac{dw}{gdt} - \frac{qU}{g}\right) &= - W\phi\cos\theta_o \\
&\quad - Z_o - wZ_w - pZ_p - qZ_q \quad . \quad . \quad . \qquad (11w) \\
A\frac{dp}{gdt} - F\frac{dq}{gdt} &= - L_o - wL_w - pL_p - qL_q \quad . \quad . \quad . \qquad (11p) \\
B\frac{dq}{gdt} - F\frac{dp}{gdt} &= - M_o - wM_w - pM_p - qM_q \quad . \quad . \qquad (11q) \\
C\frac{dr}{gdt} &= - (H_o + \delta H)h \\
&\quad - N_o - uN_u - vN_v - rN_r \quad . \quad . \quad . \qquad (11r)
\end{aligned}$$

We substitute from the equations of equilibrium and rearrange the equations in two groups, the first group containing those (the first, second, and sixth) involving u, v, r, the second (consisting of the third, fourth, and fifth) involving w, p, q. We thus obtain the groups

$$\begin{aligned}
W\frac{du}{gdt} &= W\epsilon\cos\theta_o + \delta H - uX_u - vX_v - rX_r \qquad (12u) \\
W\left(\frac{dv}{gdt} + \frac{rU}{g}\right) &= - W\epsilon\sin\theta_o \quad - uY_u - vY_v - rY_r \qquad (12v) \\
C\frac{dr}{gdt} &= - h\delta H \qquad - uN_u - vN_v - rN_r \qquad (12r)
\end{aligned}$$

and the group

$$\begin{aligned}
W\left(\frac{dw}{gdt} - \frac{qU}{g}\right) &= - W\phi\cos\theta_o - wZ_w - pZ_p - qZ_q \qquad (13w) \\
A\frac{dp}{gdt} - F\frac{dq}{gdt} &= \qquad - wL_w - pL_p - qL_q \qquad (13p) \\
B\frac{dq}{gdt} - F\frac{dp}{gdt} &= \qquad - wM_w - pM_p - qM_q \qquad (13q)
\end{aligned}$$

The first group represent oscillations in the plane of x, y, which we call **longitudinal** or **symmetrical oscillations**; the second group represent rotations p, q about the axes of x and y and motions w perpendicular to the plane of x, y, which we should describe as **lateral** or **transverse oscillations**. It will be noticed, however, that while p represents a cant over sideways, q represents a turning round of the direction of the machine, say a directional rotation, and w determines a lateral motion of translation. In view of the frequent attempts that have been made to separate "lateral" and "directional" stability, and the fact that the motions are not all rotations, it is better to distinguish the oscillations of the second group by the term **asymmetric oscillations**. Another alternative is to apply the term *lateral* to the whole group of asymmetric oscillations, which are in general mutually interdependent, and in this book it will never be used except in this sense.

We further have,

$$\phi = 0, \quad \theta = \theta_o + \epsilon, \quad \frac{d\epsilon}{dt} = \frac{d\theta}{dt} = r, \qquad \frac{d\phi}{dt}\cos\theta_o = p\cos\theta_o - q\sin\theta_o \quad (14)$$

as may easily be deduced from equations (4) or from geometrical considerations.

Symmetric Oscillations.—Condition of Longitudinal Stability.

18. To investigate the symmetric oscillations we assume u, v, r, and ϵ to be proportional to $e^{\lambda t}$, so that

$$\frac{du}{dt} = \lambda u, \quad \frac{dv}{dt} = \lambda v, \quad \frac{dr}{dt} = \lambda r, \quad \frac{d\epsilon}{dt} = \lambda\epsilon$$

and we notice that the last equation gives $r = \lambda\epsilon$; but the equations are of a more symmetrical form when r/λ is substituted for ϵ than they would be if $\lambda\epsilon$ were substituted for r (this is, however, a mere matter of convenience).

Equations (12 u, v, r) now become on transposing

$$\begin{aligned}
&\left(W\frac{\lambda}{g} + X_u\right)u + X_v r &&+ \left(-\frac{W}{\lambda}\cos\theta_o + X_r\right)r &&= \delta H\\
&Y_u u + \left(W\frac{\lambda}{g} + Y_v\right)r &&+ \left(\frac{W}{\lambda}\sin\theta_o + W\frac{U}{g} + Y_r\right)r &&= 0\\
&N_u u + N_v r &&+ \left(C\frac{\lambda}{g} + N_r\right)r &&= -h\delta H
\end{aligned} \tag{15}$$

We consider at the outset the simplest case in which the propeller thrust is independent of the velocity, so that $\delta H = 0$; in a later section we shall examine the necessary modifications when this assumption does not hold good. The right hand sides of the above equations vanish, and on eliminating u, v, r, the result assumes the form of the determinant

$$\begin{vmatrix}
W\frac{\lambda}{g} + X_u, & X_v, & -\frac{W}{\lambda}\cos\theta_o + X_r,\\
Y_u, & W\frac{\lambda}{g} + Y_v, & \frac{W}{\lambda}\sin\theta_o + W\frac{U}{g} + Y_r,\\
N_u, & N_v, & C\frac{\lambda}{g} + N_r,
\end{vmatrix} = 0 \tag{16}$$

Multiplying by λ to remove λ from the denominator in the upper line and developing the determinant in powers of λ we get an equation of the fourth degree which we write

$$\mathfrak{A}_o\lambda^4 + \mathfrak{B}_o\lambda^3 + \mathfrak{C}_o\lambda^2 + \mathfrak{D}_o\lambda + \mathfrak{E}_o = 0 \quad . \quad . \quad . \tag{17}$$

where

$$\begin{aligned}
\mathfrak{A}_o &= CW^2\\
\frac{\mathfrak{B}_o}{g} &= CW(X_u + Y_v) + W^2N_r\\
\frac{\mathfrak{C}_o}{g^2} &= C(X_uY_v - X_vY_u) + W\{(Y_vN_r - Y_rN_v) + (X_uN_r - X_rN_u)\}\\
&\quad - W^2\frac{U}{g}N_v\\
\frac{\mathfrak{D}_o}{g^3} &= X_u(Y_vN_r - Y_rN_v) + X_v(Y_rN_u - Y_uN_r) + X_r(Y_uN_v - Y_vN_u)\\
&\quad + W\frac{U}{g}(X_vN_u - X_uN_v) + \frac{W^2}{g}(N_u\cos\theta_o - N_v\sin\theta_o)\\
\frac{\mathfrak{E}_o}{g^4} &= \frac{W}{g}\{-\cos\theta_o(Y_uN_v - Y_vN_u) - \sin\theta_o(X_uN_v - X_vN_u)\}
\end{aligned} \tag{18}$$

These coefficients may be simplified with advantage by the following notation :—write Δ_0 for the determinant

$$\begin{vmatrix} X_u & X_v & X_r \\ Y_u & Y_v & Y_r \\ N_u & N_v & N_r \end{vmatrix}$$

and write u_X for the minor of X_u, and similarly for other minors, taken with their proper algebraic sign in such a way that (*e.g.*)

$$X_u u_X + X_v v_X + X_r r_X = \Delta_0 = X_u u_X + Y_u u_Y + N_u u_N, \text{ \&c.}$$

then we get

$$\begin{aligned}
\mathfrak{A}_0 &= CW^2 \\
\frac{\mathfrak{B}_0}{g} &= CW(X_u + Y_v) + W^2 N_r \\
\frac{\mathfrak{C}_0}{g^2} &= Cr_N + W(u_X + v_Y) - W^2\frac{U}{g}N_v \\
\frac{\mathfrak{D}_0}{g^3} &= \Delta_0 + W\frac{U}{g}r_Y + \frac{W^2}{g}(N_u \cos\theta_0 - N_v \sin\theta_0) \\
\frac{\mathfrak{E}_0}{g^4} &= -\frac{W}{g}(r_X \cos\theta_0 - r_Y \sin\theta_0) \qquad . \qquad . \quad . \quad . \qquad (18a)
\end{aligned}$$

19. The conditions of stability require that all the four roots of the biquadratic equation for λ shall have their real part negative. This follows from the assumption that the small disturbances u, v, r, etc., are all proportional to $e^{\lambda t}$ in a typical oscillation. If λ_1, λ_2, λ_3, λ_4 are the roots, the expressions for u, v, r, take the form

$$a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_3 e^{\lambda_3 t} + a_4 e^{\lambda_4 t}$$

a_1, a_2, a_3, a_4 being constant coefficients determined by the initial conditions.

If any one of the roots λ_1 is real and positive a disturbance of the form $u = a_1 e^{\lambda_1 t}$ will increase indefinitely with the time and steady motion will be unstable.

If on the other hand λ_1 is real and negative and equal say to $-\kappa$, the expressions $u = ae^{-\kappa t}$ represent disturbances which decrease with the time, the modulus of decay or **coefficient of subsidence** being κ. For such disturbances the steady motion will be stable.

If the biquadratic has a pair of complex roots $a \pm \beta i$ where $i = \sqrt{-1}$, the corresponding disturbance takes the form $e^{at}(a \cos \beta t + b \sin \beta t)$, and if the real part a is positive this represents an oscillation which increases with the time and steady motion will be unstable. If on the other hand the real part a is negative and equal to $-\gamma$, the solution takes the form $e^{-\gamma t}(a \cos \beta t + b \sin \beta t)$ and the disturbance becomes a **damped oscillation** of which the **modulus of decay** or coefficient of subsidence is equal to γ. For such disturbances the system tends to revert to its state of steady motion and is stable.

The condition that the roots of a biquadratic equation shall all have their real part negative and thus indicate stability of steady motion is given by Routh ("Advanced Rigid Dynamics"). Supposing that $\mathfrak{A}_o$ is positive (as is the case above) this condition requires that

$$\mathfrak{A}_o, \mathfrak{B}_o, \mathfrak{C}_o, \mathfrak{D}_o, \mathfrak{E}_o, \text{ and } \mathfrak{H}_o,$$

where

$$\mathfrak{H}_o \equiv \mathfrak{B}_o\mathfrak{C}_o\mathfrak{D}_o - \mathfrak{A}_o\mathfrak{D}_o{}^2 - \mathfrak{E}_o\mathfrak{B}_o{}^2 \quad . \quad . \quad . \quad (19)$$

shall all be positive. For a proof of these conditions we refer to Routh's treatise, but a partial verification of the last condition $\mathfrak{H}_o > 0$ may be given by examining what happens if the real part of a pair of roots of the biquadratic (17)

$$\mathfrak{A}_o\lambda^4 + \mathfrak{B}_o\lambda^3 + \mathfrak{C}_o\lambda^2 + \mathfrak{D}_o\lambda + \mathfrak{E}_o = 0$$

from being negative becomes zero. In such a case the roots $\beta\iota$, $-\beta\iota$ are roots of $\lambda^2 + \beta^2 = 0$ and by substitution and equating the real and imaginary parts we find that $\lambda^2 = -\beta^2$ must be a solution of both equations

$$\mathfrak{A}_o\lambda^4 + \mathfrak{C}_o\lambda^2 + \mathfrak{E}_o = 0$$
$$\mathfrak{B}_o\lambda^3 + \mathfrak{D}_o\lambda = 0$$

The second gives

$$\lambda^2 \text{ or } -\beta^2 = -\frac{\mathfrak{D}_o}{\mathfrak{B}_o}$$

which, substituted in the first, gives

$$\mathfrak{A}_o\frac{\mathfrak{D}_o^2}{\mathfrak{B}_o^2} - \mathfrak{C}_o\frac{\mathfrak{D}_o}{\mathfrak{B}_o} + \mathfrak{E}_o = 0$$

or

$$\mathfrak{B}_o\mathfrak{C}_o\mathfrak{D}_o - \mathfrak{E}_o\mathfrak{B}_o^2 - \mathfrak{A}_o\mathfrak{D}_o^2 = 0$$

which therefore represents the limiting case when the real part of a pair of roots changes sign. The corresponding inequality representing a condition for stability may also be written

$$\mathfrak{C}_o - \mathfrak{E}_o\frac{\mathfrak{B}_o}{\mathfrak{D}_o} - \mathfrak{A}_o\frac{\mathfrak{D}_o}{\mathfrak{B}_o} > 0 \quad . \quad . \quad . \quad . \quad (19a)$$

and necessarily implies the condition $\mathfrak{C}_o$ positive if all the other four coefficients are positive.

Asymmetric Oscillations.—Condition of Lateral Stability.

20. For the asymmetric oscillations we take w, p, q, and ϕ proportional to $e^{\lambda t}$, and notice that in this case we have

$$\lambda\phi\cos\theta_o = \frac{d\phi}{dt}\cos\theta_o = p\cos\theta_o - q\sin\theta_o$$

which we use to substitute for ϕ in terms of p and q. Our equations (13 w, p, q) now become

$$\begin{aligned}
\left(W\frac{\lambda}{g} + Z_w\right)w + \left(\frac{W}{\lambda}\cos\theta_o + Z_p\right)p + \left(-W\frac{U}{g} - \frac{W}{\lambda}\sin\theta_o + Z_q\right)q &= 0\\
L_w w + \left(A\frac{\lambda}{g} + L_p\right)p + \left(-F\frac{\lambda}{g} + L_q\right)q &= 0\\
M_w w + \left(-F\frac{\lambda}{g} + M_p\right)p + \left(B\frac{\lambda}{g} + M_q\right)q &= 0
\end{aligned} \qquad (20)$$

giving on elimination

$$\begin{vmatrix}
W\frac{\lambda}{g} + Z_w, & \frac{W}{\lambda}\cos\theta_o + Z_p, & -W\frac{U}{g} - \frac{W}{\lambda}\sin\theta_o + Z_q\\
L_w, & A\frac{\lambda}{g} + L_p, & -F\frac{\lambda}{g} + L_q\\
M_w, & -F\frac{\lambda}{g} + M_p, & B\frac{\lambda}{g} + M_q
\end{vmatrix} = 0 \qquad (21)$$

Multiplying by λ and expanding the determinant in powers of λ in the form

$$\mathfrak{A}_1\lambda^4 + \mathfrak{B}_1\lambda^3 + \mathfrak{C}_1\lambda^2 + \mathfrak{D}_1\lambda + \mathfrak{E}_1 = 0 \quad . \quad . \quad . \quad (22)$$

we now have for the coefficients

$$\begin{aligned}
\mathfrak{A}_1 &= W(AB - F^2) \\
\frac{\mathfrak{B}_1}{g} &= Z_w(AB - F^2) + W\{AM_q + BL_p + F(L_q + M_p)\} \\
\frac{\mathfrak{C}_1}{g^2} &= Z_w\{AM_q + BL_p + F(L_q + M_p)\} + W(L_pM_q - L_qM_p) \\
&\quad - Z_p(FM_w + BL_w) - \left(Z_q - W\frac{U}{g}\right)(FL_w + AM_w) \\
\frac{\mathfrak{D}_1}{g^3} &= Z_w(L_pM_q - L_qM_p) + Z_p(L_qM_w - M_qL_w) \\
&\quad + \left(Z_q - W\frac{U}{g}\right)(L_wM_p - L_pM_w) \\
&\quad + \frac{W}{g}\{(FL_w + AM_w)\sin\theta_o - (BL_w + FM_w)\cos\theta_o\} \\
\frac{\mathfrak{E}_1}{g^4} &= \frac{W}{g}\{(L_qM_w - M_qL_w)\cos\theta_o - (L_wM_p - M_wL_p)\sin\theta_o\}. \qquad (23)
\end{aligned}$$

The expression for $\mathfrak{C}_1$ may also be arranged thus :—

$$\begin{aligned}
\frac{\mathfrak{C}_1}{g^2} &= A(Z_wM_q - M_wZ_q) + B(Z_wL_p - L_wZ_p) \\
&\quad + F\{(Z_wM_p - M_wZ_p) + (Z_wL_q - L_wZ_q)\} + W(L_pM_q - L_qM_p) \\
&\quad + W\frac{U}{g}(FL_w + AM_w)
\end{aligned}$$

Writing as before

$$\Delta_1 = \begin{vmatrix} Z_w & Z_p & Z_q \\ L_w & L_p & L_q \\ M_w & M_p & M_q \end{vmatrix}$$

and using as previously, *e.g.*, w_Z to denote the minor of Z_w, we get

$$\begin{aligned}
\mathfrak{A}_1 &= W(AB - F^2) \\
\frac{\mathfrak{B}_1}{g} &= Z_w(AB - F^2) + W\{AM_q + BL_p + F(L_q + M_p)\} \\
\frac{\mathfrak{C}_1}{g^2} &= Ap_L + Bq_M - F(q_L + p_M) + Ww_Z \\
&\quad + W\frac{U}{g}(AM_w + FL_w) \\
\frac{\mathfrak{D}_1}{g^3} &= \Delta_1 - W\frac{U}{g}q_Z + \frac{W}{g}\{(A\sin\theta_o - F\cos\theta_o)M_w + (F\sin\theta_o - B\cos\theta_o)L_w\} \\
\frac{\mathfrak{E}_1}{g^4} &= \frac{W}{g}(p_Z\cos\theta_o - q_Z\sin\theta_o) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23a)
\end{aligned}$$

and if, as before,

$$\mathfrak{H}_1 = \mathfrak{B}_1\mathfrak{C}_1\mathfrak{D}_1 - \mathfrak{A}_1\mathfrak{D}_1^2 - \mathfrak{E}_1\mathfrak{B}_1^2 \quad . \quad . \quad . \quad (24)$$

the conditions of asymmetric stability require that

$$\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1, \mathfrak{D}_1, \mathfrak{E}_1, \text{ and } \mathfrak{H}_1$$

shall be positive; $\mathfrak{A}_1$ being of necessity positive, as is easily shown by the principles of elementary rigid dynamics.

Observations on the Resistance Derivatives.

21. It will now be seen that if for a given aeroplane, moving in a given manner, we knew the values of the eighteen coefficients, $X_u \ldots N_r$, or, as we may write them for shortness $(X, Y, N)_{u, v, r}$ and $(Z, L, M)_{w, p, q}$, as well as the dynamical coefficients, namely, mass and moments of inertia, we could, by substitution in the formulæ obtained above, ascertain if the motion in question were symmetrically or asymmetrically stable; also by numerical solution of the biquadratic we could ascertain the character of the small oscillations about steady motion, their periods if periodic, their logarithmic decrements corresponding to stability or logarithmic increments corresponding to instability. For the coefficients $(X, Y, N)_{u, v, r}$ and $(Z, L, M)_{w, p, q}$, I propose the name **resistance derivatives**, as it is convenient to have some name for them.

Now *assuming the air resistances to be proportional to the square of the relative wind velocity*—and this is the one assumption the validity of which (under normal conditions) is *generally* admitted—the six forces and couples X, Y, Z, L, M, N will be quadratic functions of $U+u, v, w, p, q, r$, and their derivatives $X_u \ldots N$ will be proportional to U, remembering that we are only going to a first approximation, and that squares and products of the small quantities u, v, w, p, q, r are neglected in dealing with small oscillations. The quotients $X_u/U, \ldots N_r/U$ will be constants for a particular machine flying in a particular way. They

will depend on the form, dimensions, and arrangement of the aeroplanes and of the supporting frameworks (which of course also encounter air resistance), and they will also depend on the inclination of the aeroplane to the line of flight. If they were determined experimentally for every such inclination that might occur in practice, the problem of stability would be reduced to arithmetical calculation. At the same time, even granting the necessary experimental data to be known, the calculations would be exceedingly laborious, and in order to simplify them it would still be necessary to search for approximate methods of solution by developing the problem further on the mathematical side as a first step.

In the next place, we observe that since in rectilinear motion along the axis of x, we have X, Y, N proportional to $(U+u)^2$, therefore

$$X_u = \frac{2X_0}{U} \quad , \quad Y_u = \frac{2Y_0}{U} \quad , \quad N_u = \frac{2N_0}{U} \quad . \quad . \quad . \quad (25)$$

and X_0, Y_0, N_0 are given by the conditions of equilibrium,

$$0 = W \sin\theta_0 + H_0 - X_0, \; 0 = W\cos\theta_0 - Y_0, \; 0 = \; - H_0 h - N_0$$

and in particular if the propeller thrust is central so that $N_0 = 0$ we have also $N_u = 0$.

Remembering that the derivatives are proportional to U, we find that the coefficients in the biquadratics in $\lambda\, g$ are dependent on U in the following manner, the suffixes being omitted from these coefficients as the conclusions are the same for both the "symmetrical" and "asymmetrical" biquadratic :—

$\mathfrak{A}$ is independent of U
$\mathfrak{B}$ is proportional to U
$\mathfrak{C}$,, ,, ,, U^2
$\mathfrak{D}$ is of the form $AU^3 + BU$
$\mathfrak{E}$ is proportional to U^2

We may, however, write either biquadratic in the form of an equation in λ/gU, thus :—

$$\mathfrak{A}\left(\frac{\lambda}{gU}\right)^4 + \frac{\mathfrak{B}}{U}\left(\frac{\lambda}{gU}\right)^3 + \frac{\mathfrak{C}}{g^2U^2}\left(\frac{\lambda}{gU}\right)^2 + \frac{\mathfrak{D}}{g^3U^3}\left(\frac{\lambda}{gU}\right) + \frac{\mathfrak{E}}{g^4U^4} = 0$$

and the coefficients of powers of λ/gU will then be as follows :—

$$
\begin{array}{ll}
\mathfrak{A} & \text{constant} \\
\dfrac{\mathfrak{B}}{gU} & \text{,,} \\
\dfrac{\mathfrak{C}}{g^2U^2} & \text{,,} \\
\dfrac{\mathfrak{D}}{g^3U^3} & \text{of form } A + BU^{-2} \\
\dfrac{\mathfrak{E}}{g^4U^4} & \text{proportional to } U^{-2}
\end{array}
$$

The discriminant of the equation in this form is

$$\frac{\mathfrak{H}}{U^6} = \frac{\mathfrak{B}\mathfrak{C}\mathfrak{D} - \mathfrak{A}\mathfrak{D}^2 - \mathfrak{E}\mathfrak{B}^2}{U^6}$$

and is of the form $P + QU^{-2} + RU^{-4}$, where P, Q, R are constants. The condition of stability would thus appear to impose limitations on the value of U.

It must be remembered, however, that stability, at least longitudinal, according to all accepted ideas, has no meaning unless equilibrium exists, and when this is the case we may put the conditions of stability into different other forms by combining them with the conditions of equilibrium in different ways. We thus have to distinguish between the *primitive conditions of stability* in which no use has been made of the equations of equilibrium, and *modified conditions* in which these equations have been used to effect simplifications in the conditions of stability. If, for example, we say that an aeroplane is stable when its velocity exceeds a certain limit, it must be remembered at the same time that this velocity is determined by the conditions of equilibrium, and that a *modified condition of stability* may be substituted, in which the velocity does not appear.[1]

On the other hand, there may be cases in which it is desirable to consider stability apart from the question of

[1] The discrepancies between certain results given in this book and those given in Bryan and Williams' paper are thus accounted for.

equilibrium. Thus it is important that when the motion of an aeroplane is being accelerated in the vertical plane, it should not tend to swing round sideways, in other words, that it should possess asymmetric stability even when the forces acting on it are not in longitudinal equilibrium. Of course, in such a case the resistance derivatives will really be functions of the time, so that the investigation of the small oscillations will really be of a complicated character. It may perhaps, however, be sufficient to assume that the primitive conditions of stability, determined on the hypothesis of steady motion, hold good at every instant of the motion.

Stability of a dirigible.

22. If it be desired to investigate the stability of a dirigible balloon by the methods of the present monograph several modifications will have to be made.

In the first place account will have to be taken of the inertia of the displaced air. In the case of a sphere moving in a perfect liquid we know that this inertia is equivalent to adding half the mass of the displaced liquid to the mass of the sphere. For a balloon in equilibrium the total weights of the airship and the displaced air are equal, and hence the inertia of the air becomes of considerable importance. The inertia of the contents of the gas-bag must also be taken into account. If we suppose the balloon moving in air to be represented by an ellipsoid moving in an incompressible medium, we can find an expression for the kinetic energy by hydrodynamical methods. This expression will be, owing to the (assumed) symmetry of the machine about a vertical plane, the sum of two homogeneous quadratic functions, one of $U+u, v, r$, and the other of w, p, q, and the equations of motion can be written down by Hayward's method.

In the second place the effects of gravity will be entirely different, the weight of the airship being balanced by the buoyancy of the air according to the principles of Archimedes. In a state of equilibrium (at rest) the centres of gravity and buoyancy are in the same vertical line. If c is the distance between them, then when the axis of x is been depressed from a horizontal position through an angle θ, and the machine subsequently turned through an angle ϕ about this axis, couples are produced about the axes of

	x	y	z
of amount	$-cW\cos\theta\sin\phi$	0	$-cW\sin\theta$

and there are no component forces due to gravity along the axes. *Consequently the terms depending on the action of gravity occur in the equations of rotation about the axes of* x *and* z *instead of in the equations of translation, and the character of the oscillations is thus entirely different.*

The principal obstacle in the way of a satisfactory treatment of the stability of a dirigible is the difficulty of making suitable assumptions regarding air resistances due to causes other than the inertia already mentioned. We might, of course, assume that these effects were, as in the case of aeroplanes, represented to the first order by terms of the form

$$-X_o - uX_u - vX_v - rX_r$$

where X_o is proportional to U^2 and the rest are proportional to U. But to suppose that it is legitimate merely to *add* these terms to those dependent on the inertia of the displaced air (which depend on the accelerations of the system) is a very shaky assumption. All that could be said is that it would lead to conclusions which might be regarded as a basis of comparison with results of experiment.

CHAPTER III

GENERAL CONSIDERATIONS REGARDING SYMMETRICAL DERIVATIVES

Expressions for Plane Areas.

23. We shall now show how in the case of *plane* supporting surfaces (setting aside camber for the present) the nine resistance derivatives $(X, Y, N)_{u, v, r}$ can be found if the normal resistance and the position of the centre of pressure are known functions of the angle of attack and of the angular velocity r. In the first place we assume, for simplicity, that there is no tangential resistance; such a resistance may be considered subsequently.

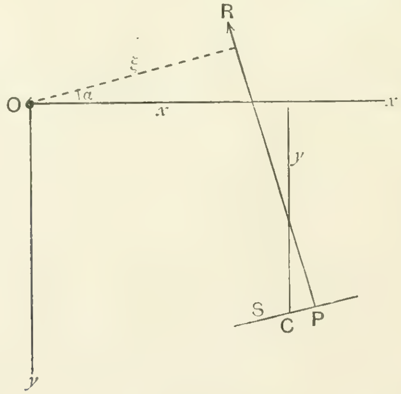

FIG. 4.

Let S be a plane area perpendicular to the plane of (x, y) and slanting upwards at an "*angle of attack*" a

to the axis of x. Let C be a point whose coordinates are x, y, centrally situated on S, which may be taken in the first instance to be an origin from which the distances of the centre of pressure and other points on the plane are measured; P the centre of pressure. Let R be the resultant thrust, ξ the perpendicular distance of its line of action from O. Then the component forces and couple due to resistance (taken positive when retarding motion) are

$$X = R \sin a \qquad Y = R \cos a \qquad N = R\xi \quad . \quad . \quad (26)$$

The angle of attack being a, we may write

$$\begin{aligned} R &= KSU^2 f(a) \\ CP &= a\,\phi(a) = s \text{ (suppose)} \end{aligned} \quad . \quad . \quad . \quad (27)$$

where K is the coefficient of resistance,

a is the semi breadth of the plane, or a quantity determined by its linear dimensions,

$f(a)$, $\phi(a)$ are functions determined by experiment, representing the relations connecting the resultant thrust, and position of the centre of pressure with the angle of attack.

We have further

$$\xi = x \cos a - y \sin a + a\,\phi(a) = x \cos a - y \sin a + s \quad . \quad (28)$$

If now the additional velocity components u, v, r are impressed, the velocity-components of C will be

$$U + u - yr \quad \text{and} \quad v + xr$$

and the angular velocity of the lamina about C will be r.

To the first order of the small quantities u, v, r, the effect of these added velocities is to increase the resultant velocity of C to $U+u-yr$ (as is evident by neglecting the second term in the exact expression:

$$\text{vel.}^2 = (U + u - yr)^2 + (v + xr)^2)$$

and to alter the direction of the velocity of C, thus

increasing the angle of attack relative to C by an amount

$$\tan^{-1}\frac{v + xr}{U + u - yr} = \frac{v + xr}{U}$$

radians to the first order of small quantities.

Hence, to the first order the new values of R and the distance ξ, are, *so far as they depend on the translational velocity-components of C*, given by

$$R = KSU^2 f(a) + 2KSU(u - yr)f(a) + KSU(v + xr)f'(a) \quad . \quad (29)$$

$$\xi = x \cos a - y \sin a + a\,\phi(a) + a\,\frac{v + xr}{U}\,\phi'(a) \quad . \quad . \quad . \quad (30)$$

where $f'(a)$ and $\phi'(a)$ are the differential coefficients of $f(a)$ and $\phi(a)$.

24. But the lamina possesses in addition an angular velocity r about C, and in consequence of this neither the resultant thrust R, nor the coordinate of the centre of pressure can be expected to be the same as if the motion of the lamina were one of uniform translation. To allow for this difference, it is necessary and sufficient to assume that $f(a)$ and $\phi(a)$ not only are functions of the angle of attack a relative to the point C, but also that they depend on the angular velocity r of the lamina relative to C.

Now, if the velocity U and the angular velocity r are increased in any given ratio $n:1$, it is clear that the motion of the plane will be the same as previously, but will take place n times as quickly. The resultant thrust will be n^2 times as great, being proportional to U^2, and the position of the centre of pressure will be the same as before. From this reasoning we see that $f(a)$ and $\phi(a)$ must be regarded as functions of the ratio r/U as well as of the angle of attack a relative to the point C, and we may, therefore, write their corresponding variations to the first order

$$\frac{r}{U}\,f_r(a) \text{ and } \frac{r}{U}\,\phi_r(a)$$

so that

$$f_r(a) = U\frac{d}{dr}\, f(a), \quad \phi_r(a) = U\frac{d}{dr}\, \phi(a)$$

where $f_r(a)$ and $\phi_r(a)$ are certain coefficients which we may call the *rotary derivatives* of $f(a)$ and $\phi(a)$. The physical assumption that is here made is merely the generalisation of the law according to which the resistances vary as the square of the velocity, so that if all the parts of a system move n times as fast, the forces due to air resistance are increased n^2 times without being changed in direction or position.

25. Making the necessary substitutions we obtain the following results, writing $\xi = x \cos a - y \sin a + a\phi(a)$.

$$\frac{X_o}{KSU^2} = f(a)\sin a, \quad \frac{X_u}{KSU} = 2f(a)\sin a, \quad -\frac{X_v}{KSU} = f'(a)\sin a,$$

$$\frac{X_r}{KSU} = \{xf'(a) - 2yf(a) + f_r(a)\}\sin a$$

$$-\frac{Y_o}{KSU} = f(a)\cos a, \quad \frac{Y_u}{KSU} = 2f(a)\cos a, \quad \frac{Y_v}{KSU} = f'(a)\cos a,$$

$$\frac{Y_r}{KSU} = \{xf'(a) - 2yf(a) + f_r(a)\}\cos a$$

$$\frac{N_o}{KSU^2} = f(a)\xi, \quad \frac{N_u}{KSU} = 2f(a)\xi, \quad \frac{N_v}{KSU} = f'(a)\xi + af(a)\phi'(a),$$

$$\frac{N_r}{KSU} = \{xf'(a) - 2yf(a) + f_r(a)\}\xi + f(a)\{xa\phi'(a) + a\phi_r(a)\} \tag{31}$$

If there are two or more planes and the actions of the air pressures on them are independent of each other, the derivatives for the system are obtained by adding those for the separate planes. This assumes that the air encountered by one of the planes has not been disturbed or set in motion by the other, in other words, that neither plane comes into the "wash" caused by the other. The effects of "wash" will be discussed briefly at the end of the discussion on longitudinal stability, although a great deal of experimental work would be necessary to attain anything like finality in dealing with them.

26. We now notice the following results which greatly simplify the subsequent calculations. These results,

unfortunately, did not come to my notice until much long and laborious algebra had been gone through, by which they were proved in the first instance.

(i) *If there is only a single plane surface, or even if there are several parallel planes, the determinant* Δ_o *vanishes and in addition the minors of its third line* u_N, v_N, r_N *vanish.*

(ii) *If there are two plane surfaces only, and if the displacements of the centre of pressure,* $a\phi'(a), a\phi_r(a)$, *be neglected,* then the determinant of the derivatives for either plane has *three* rows and *three* columns proportional to one another, *i.e.*, using accented and unaccented letters to refer to the two planes the determinants,

$$\begin{vmatrix} X_u & X_v & X_r \\ Y_u & Y_v & Y_r \\ N_u & N_v & N_r \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} X'_u & X'_v & X'_r \\ Y'_u & Y'_v & Y'_r \\ N'_u & N'_v & N'_r \end{vmatrix}$$

each have three rows and three columns proportional, and the resulting determinant Δ_o, which is

$$\begin{vmatrix} X_u + X'_u & X_v + X'_v & X_r + X'_r \\ Y_u + Y'_u & Y_v + Y'_v & Y_r + Y'_r \\ N_u + N'_u & N_v + N'_v & N_r + N'_r \end{vmatrix}$$

can be expressed as the sum of eight determinants, one formed by taking the first determinant, three formed by taking two rows or columns of the first and the remaining row or column from the second, and so on. In such cases again, each of these determinants vanishes since it has at least two rows or columns proportional to each other, and the result is that Δ_o *again vanishes.*[1]

---

[1] An alternative proof is given as follows. Writing $\xi'' = xf''(a) + 2yf(a) + f_r(a)$ and using suffixes 1, 2 to refer to the two planes, the determinant Δ_o is immediately recognised as the product (written down by the ordinary rule) of two determinants, namely:

$$U^3 \begin{vmatrix} 2K_1S_1f(a_1) & 2K_2S_2f(a_2) & 0 \\ K_1S_1f'(a_1) & K_2S_2f'(a_2) & 0 \\ K_1S_1\xi_1'' & K_2S_2\xi_2'' & 0 \end{vmatrix} \times \begin{vmatrix} \sin a_1 & \sin a_2 & 0 \\ \cos a_1 & \cos a_2 & 0 \\ \xi_1 & \xi_2 & 0 \end{vmatrix}$$

which is equal to zero.

(iii) If $\phi'(a)$ and $\phi_r(a)$ be not neglected, and there are two plane surfaces, the only parts of Δ_o which do not vanish will be those in which $\phi'(a_1)$, $\phi'(a_2)$, $\phi_r(a_1)$, $\phi_r(a_2)$ occur as factors, a_1, a_2 being the inclinations of the two planes, and these will be multiplied into the minors v_N, r_N; and it is easy to see

$$r_N = 2U^2K_1K_2S_1S_2\{f(a_1)f'(a_2) - f(a_2)f'(a_1)\} \sin(a_1 - a_2) \quad . \quad (32)$$

and that v_N can be expressed in a somewhat similar form, also containing $\sin(a_1 - a_2)$ as a factor. Without looking further at the present stage of the investigation, it will be seen that the determinant Δ_o, so far from giving rise to laborious calculations, either vanishes or reduces to something readily calculable and probably in *most* practical applications negligible. An exception will be discussed in due course, in which it will be shown that in some cases stability may be increased by the use of *three* planes especially arranged so as purposely to give a value of Δ_o different from zero.

(iv) In the case of a single plane surface, using the expression for $\mathfrak{E}_o$, namely,

$$\frac{\mathfrak{E}_o}{g^4} = -\frac{W}{g}\{r_X \cos\theta_o - r_Y \sin\theta_o\}$$

we obtain

$$\frac{\mathfrak{E}_o}{g^4} = -\frac{2W}{g}K^2S^2U^2\{f(a)\}^2 a\phi'(a)\cos(a - \theta_o) \quad . \quad . \quad (33)$$

and it follows from the condition $\mathfrak{E}_o$ positive that *a single plane cannot be stable unless $\phi'(a)$ is negative, that is unless the centre of pressure moves forward when the angle of attack decreases.* This is the case in actual practice. Also we see that to secure stability, the plane must be sufficiently broad for the shift of the pressure to have an appreciable effect. In such cases, however, it becomes necessary to take account of the effects of the coefficients $f_r(a)$ and $\phi_r(a)$ in the other conditions of stability. They do not occur in $\mathfrak{E}_o$.

Observations on the Rotary Derivatives.

27. In the foregoing work we have introduced two derivatives, $f_r(\alpha)$ and $\phi_r(a)$, such that $f_r(a)r/U$ and $\phi_r(a)r/U$, represent the changes in the values of $f(a)$ and $\phi(a)$ when the lamina receives a small rotation r about the point (x, y). It is important that the effects of rotation on the resultant thrust and position of centre of pressure should be investigated experimentally if broad aeroplanes and aerocurves are to be used in aerial navigation. In the original results in Bryan and Williams' paper, these **rotary derivatives,** as they may be called,

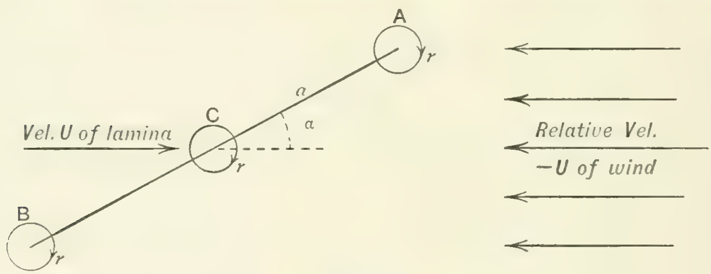

FIG. 5.

were neglected, failing the existence of such experimental data, and the same thing has certainly been done in all stability investigations till now. We may certainly get rid of one of these by suitably choosing the point C (x, y). For $f_r(a)$ only occurs in $xf''(a) - 2yf'(a) + f_r(a)$, and all that is necessary is to write

$$xf''(a) - 2yf'(a) + f_r(a) = x'f''(a) - 2y'f'(a)$$

this gives one relation determining the co-ordinates (x', y') of C_1, the new position of C, and we may further assume that C_1 is in the plane of the lamina itself. The point C_1 is characterised by the property that rotation about C_1 does not affect the resultant thrust on the lamina.

Now let AB (Fig. 5) represent the lamina, C its

middle point. Then if the lamina receive an angular velocity r about A it will move away from the wind, and we naturally infer that the resultant thrust will be decreased. If it rotate about B in the same direction it will move towards the wind, and we infer that the resultant thrust will be increased. We are thus led to believe that C_1 lies between A and B. If the lamina rotate about its centre C, the thrust will be increased on AC and decreased on BC, but as the thrust is greater on the forward part AC than on the backward part BC, it appears probable that the net result is to increase the thrust. Hence we infer that the required point C_1 is probably in front of C and quite near the centre of pressure.

A somewhat similar line of argument may be applied in connection with $\phi_r(a)$. A positive rotation r about a point on AB produced considerably beyond B will certainly increase the angle of attack and cause the centre of pressure to move *backwards*. A similar rotation about a point on BA produced considerably beyond A will decrease the angle of attack and shift the centre of pressure *forwards*. But a rotation about C appears likely to increase the pressure on the forward part of the plane and decrease that on the backward part, shifting the centre of pressure *forwards*, and hence it would appear that the point C_2 about which the lamina must be rotated in order that there may be no shifting of the centre of pressure is probably behind C, and does not coincide therefore with C_1.

The last argument is somewhat indefinite, as it is impossible to be quite sure that a current of air impinging on a rotating lamina would behave in exactly the way that appears most plausible. There are, however, two methods of investigating the question experimentally; a much more satisfactory alternative.

28. **The Whirling-table Method.**—Suppose the lamina

attached to a rotating arm and whirled in such a way as to give a forward velocity U at the point C.

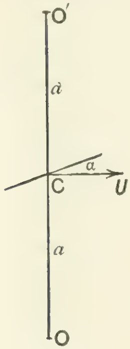

Fig. 6.

If the lamina be rotated about O where $OC = a$, then we have $r = U/a$; if rotated about a point O' on the opposite side we should have $r = -U/a$. If then $f(\alpha)$ and $\phi(\alpha)$ be observed by determining experimentally the resultant thrust and position of the centre of pressure when rotated first about O and secondly about O', the difference of the two values of $f(\alpha)$ multiplied by the length $a/2$ will give $f_r(\alpha)$, and the difference of the two values of $\phi(\alpha)$ multiplied by $a/2$ will similarly give $\phi_r(\alpha)$, the values being in either case referred to the point C.

29. **The Pendulum Method.**—According to the Blue Book for 1909–1910, experiments in a wind tunnel which produces a current of air that may be assumed uniform give better results than whirling table experiments. To adapt this method to the rotary derivatives, it is necessary

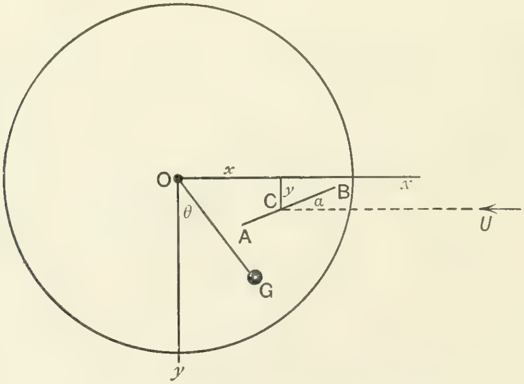

Fig. 7.

to attach the lamina under investigation to a pendulum and observe the oscillations. To eliminate the effects of air resistance on the pendulum the latter would have

to be made in the form of a disc or sphere, unequally loaded, but having its axis in the axis of rotation (Fig. 7). Then only tangential resistances would occur, the resultant couple due to these would be proportional to the square of the angular velocity (as may easily be shown), and its effect on small oscillations might be neglected for an approximate calculation.

Let AB be the lamina, O the centre of suspension, G the centre of gravity of the whole mass, θ = inclination of OG to vertical, a = inclination of AB to horizon, so that $\theta - a$ = constant. If N is the moment of the resistances about O, then when the lamina is at rest we have as in (31), § 25,

$$N = KSU^2 f(a)(x \cos a - y \sin a + a\phi(a)) \quad . \quad . \quad (34)$$

The general equation of motion is

$$I\frac{d^2\theta}{gdt^2} = -W.OG \sin\theta + N$$

where I = moment of inertia of whole mass about O.

For equilibrium we have

$$0 = -W.OG \sin\theta_o + N_o$$

In a small oscillation if $\theta = \theta_o + \epsilon$ and $a = a_o + \epsilon$, and the angular velocity $d\epsilon/dt$ is equal to $-r$, we have

$$\sin\theta = \sin\theta_o + \epsilon \cos\theta_o$$
$$N = N_o + \epsilon\frac{dN}{da} + rN_r$$

and the equation of small oscillations becomes

$$I\frac{d^2\epsilon}{gdt^2} + N_r\frac{d\epsilon}{dt} + \epsilon\left\{W.OG\cos\theta_o - \frac{dN}{da}\right\} = 0 \quad . \quad . \quad (35)$$

whence the modulus of decay is $gN_r/2I$, and if the period is $2\pi/p$, then

$$p^2 = \frac{g}{I}\left\{W.OG\cos\theta_o - \frac{dN}{da}\right\} - \frac{N_r^2 g^2}{4I^2}$$

This holds if the motion be oscillatory, a condition that can be always secured by making OG sufficiently great.

The value of dN/da is got by directly differentiating N with respect to a.

It follows that *by observing the modulus of decay* N_r *can be found, or if* N_r *be negative the logarithmic increment of the oscillations will determine it.*

Now using the notation of (31) § 25 we see that if N_r is thus found for two different positions of the point C (x, y) on the lamina, with the same angle of attack, a, the formulæ for N_r in (31) will give sufficient data to find both $f_r(a)$ and $\phi_r(a)$. It may perhaps be safe to omit further details of the method as the reader can probably supply them.

This conclusion suggests, however, the possibility of going further, and instead of using a single lamina, experimenting with a model aeroplane so as to find the values of N_r relative to different points in its plane of symmetry as centres of rotation. The idea suggests itself, to use the results to determine the resistance derivatives of the aeroplane considered as a whole, instead of building them up from a consideration of the separate surfaces. The problem thus arises: Given the values of N_r referred to different origins in the plane of symmetry, to find as much information as possible about the other resistance derivatives referred to a single origin. This involves consideration of the transformation-formulæ for resistance derivatives.

Formulæ of Transformation.

30. Let $U + u$, v, r be the velocity components, X, Y, N the resistance components referred to the origin (O); let accented letters refer to parallel axes through the point (x, y). We have

$$u = u' + yr, \quad v = v' - xr, \quad r' = r$$
$$X' = X, \quad Y' = Y, \quad N' = N - xY + yX \quad . \quad . \quad (36)$$

and the transformation formulæ fall under the following types :—

(1) For $(X, Y)_{o, u, v}$ the type $X'_{u1} = X_u$

(2) For $(X, Y)_r$ the type

$$X'_{r1} = X_r + yX_u - xX_v$$

(3) For $N_{u,v}$ the type

$$N'_{u1} = N_u + yX_u - xY_u$$

(4) For $N_r'{}_1$ the type

$$\begin{aligned} N'_r &= \left(\frac{d}{dr} + y\frac{d}{du} - x\frac{d}{dv}\right)(N + yX - xY) \\ &= N_r - x(N_v + Y_r) + y(N_u + X_r) \\ &\quad + x^2Y_v - xy(X_v + Y_u) + y^2X_u \qquad (37) \end{aligned}$$

It will thus be seen that determinations of the resistance-couple-derivative N_r about different origins will give the values of

$$X_u,\ Y_v,\ N_r,\ N_v + Y_r,\ N_u + X_r,\ Y_u + X_v$$

and as the values of X_u, Y_u, N_u are equal to $2X_o/U$, etc., only one further datum is needed. If the condition $\Delta_o = 0$ is satisfied this datum is available.

31. The formulæ for change of axes, the origin being constant, are as follows: using unaccented letters to refer to old axes and accented letters to refer to axes making an angle A, in the positive direction with them we have

$$r = r' \qquad N' = N \qquad \frac{d}{dr'} = \frac{d}{dr}$$

$$u = u' \cos A - v' \sin A, \qquad v = u' \sin A + v' \cos A$$

$$X' = X \cos A + Y \sin A \qquad Y' = - X \sin A + Y \cos A$$

$$\frac{d}{du'} = \cos A \frac{d}{du} + \sin A \frac{d}{dv} \qquad \frac{d}{dv'} = - \sin A \frac{d}{du} + \cos A \frac{d}{dv} \qquad (38)$$

whence the resistance derivatives referred to the new axes can easily be expressed in terms of those referred to the old axes, for example :

$$\begin{aligned} X'_{u'} = \frac{d}{du'} X' &= \left(\cos A \frac{d}{du} + \sin A \frac{d}{dv}\right) (X \cos A + Y \sin A) \\ &= X_u \cos^2 A + (X_v + Y_u) \sin A \cos A + Y_v \sin^2 A \end{aligned}$$

Camber.

32. **The chief difference between plane and curved surfaces** is that in the latter the direction of the resultant pressure may vary with the angle of attack, and we should have to use two letters, a, a', the first denoting the inclination of, say, the chord of the aerocurve to the axis of x, the other a' denoting the inclination of the resultant thrust to the axis of y. For Turnbull's doubly curved surfaces experimental data would alone be available, but if the section of a surface is a circular arc of radius c, it is clear that the resultant thrust will pass through the centre of curvature, and if the centre of pressure be shifted forward through a distance $\delta s = a\phi'(a)\delta a$, the direction of R will change by an amount $\delta a'$ equal to $-\delta s/c$ or $\delta a' = \delta a \times a/c\{-\phi'(a)\}$ (for a plane surface $\phi'(a)$ is negative). The most important effect of this when the angles are small will be to produce a variation δX in the horizontal component equal to $R \cos a \; \delta a$, and, when the necessary substitutions are made, it is seen that the *principal* effect of curvature is to introduce additional terms into the values of X_v and X_r. The magnitude of these terms depends on a/c, that is, on the *angle* which the arc subtends at its centre of curvature, not on the actual *size* of the aerocurve. They might thus become important even with narrow aerocurves if this angle is not small. The corrections thus introduced are discussed in § 69. Further corrections may be examined at a future time if deemed advisable.

The principal difficulty arises from our want of information regarding the rotatory coefficients $\phi_r(a)$ which also lead to variations in the angle a'; till these variations have been studied the stability of aerocurves must be to some extent uncertain, and the extension of results calculated for plane areas must be regarded as a

working hypothesis, divergences between theory and experiment being anticipated.

Narrow Planes at Small Angles.

33. Owing to the doubtful points raised in the previous paragraphs, as well as for reasons of simplicity, the study of aeroplane stability is best commenced by the consideration of systems of *narrow planes at small angles.*

By the term *narrow planes* applied to an aeroplane with two or more plane surfaces, one behind the other, it is to be implied that the breadth or "chord" of either plane is small in comparison with their horizontal distance apart. If the aeroplane receives a small angular velocity r, the difference of velocity set up between the front and rear edge of a plane will be small compared with the difference in velocity set up between the two planes, and for this reason, *the rotary derivatives may be neglected* (in comparison with quantities actually taken into account). Further it may be assumed in the first instance that *the shifting of the centres of pressure with varying angles of attack* [*i.e.*, the quantities $a\phi'(a)$] *are negligible* owing to the smallness of a.

We shall now assume that *the point C* (of § 23) the position of which has hitherto been left arbitrary *is chosen to coincide with the centre of pressure* when the aeroplane is flying steadily. That is, *we now define* x, y *to be the coordinates of the centre of pressure in steady flight,* so that $\phi(a) = 0$. The assumption that $f_r(a)$ and $a\phi_r(a)$ are negligible thus becomes equivalent to neglecting the distances from the centre of pressure of the points C_1, C_2 considered in § 27, an assumption which, as we have seen, is *probably* fulfilled more approximately in the case of C_1 than of C_2.

When the planes are *gliding at small angles* we imply that the angles of attack a are small, and in such

circumstances it will be assumed that *the normal thrust on a plane varies as the sine of the angle of attack.* This property is commonly expressed by the formula

$$P = 2P_{90} \sin a$$

where P_{90} is the pressure when the angle of attack is $90°$; of course the formula fails when a is considerable, for with $a = 90°$ it would give $P = 2P_{90}$ instead of $P = P_{90}$; other formulæ have been proposed which get over the difficulty, such as Duchemin's

$$P = 2P_{90} \frac{\sin a}{1 + \sin^2 a}$$

but for small values of a, the simpler one is usually regarded as sufficient. *We therefore write* $f(a) = \sin a$.

We further assume that $\sin a$ is small and that the approximations $\sin a = a$ and $\cos a = 1$ may be used if desired, and that terms of the order a may be neglected *in comparison with finite quantities.* Remembering, however, that in horizontal flight

$$H = \Sigma KSU^2 \sin^2 a \quad . \quad . \quad . \quad . \quad . \quad (39)$$

it is evident that even $\sin^2 a$ cannot be neglected except in comparison with quantities of lower order.

With these assumptions, and writing

$$\xi' = x \cos a - y \sin a, \quad \xi'' = x \cos a - 2y \sin a$$

the table of derivatives in § 25 reduces to

$$\begin{array}{llll}
\dfrac{X_o}{U^2} = \Sigma KS \sin^2 a & \dfrac{X_u}{U} = 2\Sigma KS \sin^2 a, & \dfrac{X_v}{U} = \Sigma KS \sin a \cos a, & \dfrac{X_r}{U} = \Sigma KS\xi'' \sin a \\
\dfrac{Y_o}{U^2} = \Sigma KS \sin a \cos a & \dfrac{Y_u}{U} = 2\Sigma KS \sin a \cos a, & \dfrac{Y_v}{U} = \Sigma KS \cos^2 a, & \dfrac{Y_r}{U} = \Sigma KS\xi'' \cos a \\
\dfrac{N_o}{U^2} = \Sigma KS\xi' \sin a & \dfrac{N_u}{U} = 2\Sigma KS\xi' \sin a, & \dfrac{N_v}{U} = \Sigma KS\xi' \cos a, & \dfrac{N_r}{U} = \Sigma KS\xi'\xi''.
\end{array} \quad (40)$$

The equations of equilibrium in horizontal flight become

$$H = U^2\Sigma KS \sin^2 a, \quad W = U^2\Sigma KS \sin a \cos a, \quad -Hh = U^2\Sigma KS\xi' \sin a \quad (41)$$

34 **The S', μ notation.**—A further simplification which will be found to be of great use, especially in dealing

with stability under complicated conditions, consists in the substitutions $S' = S \cos^2 \alpha$ and $\mu = \tan \alpha$. With this substitution the foregoing formulæ give:

$$\begin{array}{llll}
\frac{X_o}{U^2} = \Sigma KS'\mu^2 & \frac{X_u}{U} = 2\Sigma KS'\mu^2 & \frac{X_v}{U} = \Sigma KS'\mu & \frac{X_r}{U} = \Sigma KS'\mu(x - 2y\mu) \\
\frac{Y_o}{U^2} = \Sigma KS'\mu & \frac{Y_u}{U} = 2\Sigma KS'\mu & \frac{Y_v}{U} = \Sigma KS' & \frac{Y_r}{U} = \Sigma KS'(x - y\mu) \\
\frac{N_o}{U^2} = \Sigma KS'\mu(x - y\mu) & \frac{N_u}{U} = 2\Sigma KS'\mu(x - y\mu) & \frac{N_v}{U} = \Sigma KS'(x - y\mu) & \frac{N_r}{U} = \Sigma KS'(x - y\mu)(x - 2y\mu
\end{array} \tag{40a}$$

with

$$H = U^2\Sigma KS'\mu^2 \qquad W = U^2\Sigma KS'\mu \qquad -Hh = U^2\Sigma KS'\mu(x - y\mu) \tag{41a}$$

The Principle of Independence of Height.

35. Since the angles α are small, ξ' and ξ'' are both approximately equal to x, and the resistance derivatives are to this approximation independent of the y coordinates of the planes. We thus see that *if an aeroplane is stable when the planes are on a level with the centre of gravity, the stability is only very slightly affected by raising or lowering the planes.*

If instead of raising or lowering a plane vertically it be displaced in a direction *perpendicular to itself*, the value of ξ' will be unaltered, while ξ'' will be decreased by $y \sin \alpha$, so that only this small difference between ξ' and ξ'' will affect the stability, and the conditions of equilibrium will be unaffected.

In this case the line of action of the normal resultant thrust on the plane is unaffected by the displacement, and the only difference is that the *tangential* velocity of the plane due to a rotation, r, of the aeroplane is altered, the corresponding normal velocity being the same as before. When the angles of attack are small, this change of tangential velocity has only a small effect on the resultant thrust of the air, and is far less important than a change of normal velocity would be.

The present deduction may be called the "*Principle of Independence of Height.*" This property seems at first sight at variance with what one would naturally expect from analogy with the case of a pendulum whose oscillations depend essentially on the height of the point of suspension above the centre of gravity, or again from the case of an aerostat. It must, however, be remembered that the principle is based on the assumption that *tangential and head resistances are neglected*, and the difference will be made clear by taking moments in each case about the centre of gravity, with the aid of the following diagrams :—

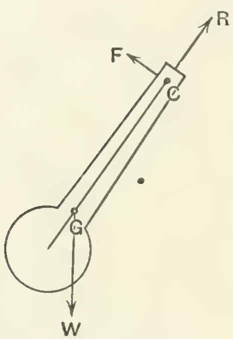

FIG. 8.

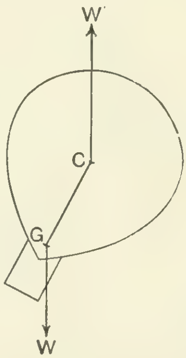

FIG. 9.

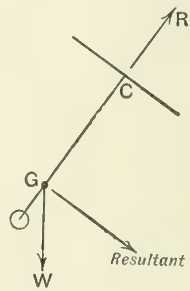

FIG. 10.

In the *pendulum* (Fig. 8), the point of suspension C is fixed so that the pendulum can only turn about C. In this case rotation about G is set up by the moment of the tangential force F acting at C at right angles to GC.

In the *aerostat* (Fig. 9), the hydrostatic thrust W', equal to the weight of the air displaced, acts vertically upwards although the centre of buoyancy C and its moment about G tends to bring GC into the vertical position by producing rotation about G.

In the *plane parachute* (Fig. 10) without tangential

resistance, if variations in the position of the centre of pressure are neglected, the resultant thrust R always passes through G, and has no moment tending to turn the parachute about G. Of course the weight W has a moment about C, but all the forces pass through G and their only effect is to produce an acceleration of the parachute as a whole in the direction of the resultant force. When the forces on a body have no moment about the centre of gravity they have no tendency to produce rotation. On the other hand, a tangential resistance would tend to turn the parachute about G, bringing the axis GC towards the vertical position.

Fig. 11 shows an aeroplane with two narrow planes, A, B, and centre of gravity, G. If the normals to the two

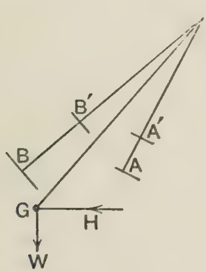

FIG. 11.

planes intersect in I, the point I will present the nearest analogy to the centre of suspension of the pendulum of Fig. 8, or centre of buoyancy of the balloon in Fig. 9. The resultant of the two normal thrusts at A and B will pass through I, and if it does not act along GI, its moment about G will tend to rotate the aeroplane. But if A or B are displaced perpendicular to themselves to A' and B', the normals to them will still meet in the same point I, and the equation of moments about I will be the same as before. The thrusts on A' and B' will also be the same as on A, B when the aeroplane has no rotation about G,

and when it rotates the normal velocities of A', B' will be the same as of A, B, while the difference of tangential velocity between A and A' only has a small influence on the resultant thrusts.

In actual practice it is certain that, owing to head resistances and tangential forces, stability will be affected to a sensible extent by raising or lowering the planes, and also, if the planes are too high, a sudden gust of wind will cause the aeroplane to heel over.

The inference, then, is that this effect must be in large measure attributed to tangential forces or head resistances.

On the other hand, the principle receives confirmation from the fact that successful flights have been made with machines in which the centre of gravity is *above* the aeroplanes.

It must not be forgotten, too, that we are now only considering systems of *planes*, not *curved* surfaces.

An important corollary is that there is no practical difference between monoplane and biplane machines in respect of stability. It is the horizontal, not the vertical, disposition of the planes on which stability mainly depends. The nomenclature "monoplane" and "biplane" does not cover the necessary distinctions. The differences with which we are concerned are between the *single-lifting system* in which the whole of the weight is supported either by one plane or by two or more vertically superposed planes, the remaining planes taking the form of a tail or rudder only, and *double-* or *multiple-lifting systems* in which the weight is supported by two or more planes (or sets of superposed planes) disposed in a fore and aft direction.

CHAPTER IV.

GRAPHIC STATICS OF LONGITUDINAL EQUILIBRIUM.

Single-Lifting Systems.

36. Before considering stability further it is useful to discuss the problem of longitudinal equilibrium with special reference to steering in a vertical plane. This can best be done by graphic methods.

In the following discussions we shall in general suppose the aeroplane to be longitudinally stable. In such cases a variation in the direction of the rudder plane or in the propeller thrust will cause the aeroplane soon to assume the new position of equilibrium determined by the altered conditions. If the aeroplane is not stable we can still examine the effects of a change of the kind on the conditions of equilibrium, although the *initial* effect of the change may be the reverse of that needed to bring the aeroplane into equilibrium and the new equilibrium position may only be attainable by careful manœuvring.

Case 1.—**Single-Lifting System with a Neutral Tail or Rudder.**—We speak of the tail or rudder as *neutral* when its plane is parallel to the direction of the relative wind so as to encounter no normal thrust. This is sometimes referred to as "grazing incidence." In this case the following three forces have to be in equilibrium :—

1. The weight W acting vertically through the centre of gravity G.
2. The resultant thrust R on the supporting plane.
3. The thrust H of the propeller.

The first condition of equilibrium is that these three forces must pass through one point.

Suppose H *passes through the centre of gravity* G, then R will also pass through G (Fig. 12).

The other condition of equilibrium is that the three forces W, R, H are parallel and proportional to the three sides of a triangle. Also W is constant while the angle

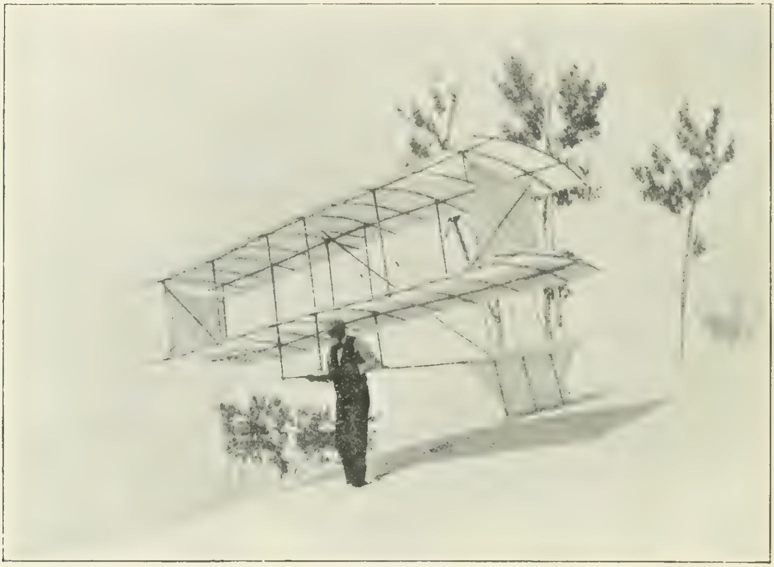

FIG. I.—A CHANUTE GLIDER.

The two superposed planes constitute a "single lifting system," the tail plane being in general "neutral" (§ 36). Being used as a glider, the longitudinal stability would be greater than in a similar aeroplane propelled horizontally (§ 55). The photograph was given to the author by the late Mr. Octave Chanute.

between H and the main plane is constant and we will denote this angle by ω. We then have the following geometrical construction for the triangle of forces:—

Draw DE vertical and of length W units. On DE describe a segment of a circle containing an angle $90 - \omega$, and in it place a line DF containing H units of length.

Then FD will be parallel to the force H and EF will be parallel and proportional to the normal force R (Fig. 13).

The aeroplane will therefore be in equilibrium when the axis of the propeller is parallel to FD and the main plane is perpendicular to EF. In order that FD and therefore H may be horizontal we must have $H = W \tan \omega$.

The direction of the aeroplane relative to the horizon is thus determined by the magnitude of the thrust H. If H be increased F will be lowered relative to D and the aeroplane will point more upwards.

The actual direction of flight is in the present instance

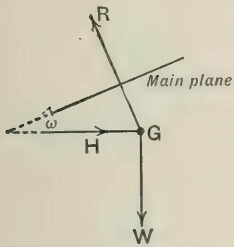

Fig. 12.

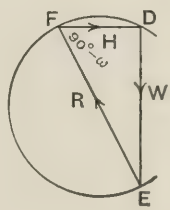

Fig. 13.

parallel to the tail or rudder plane; if this be altered in direction for steering purposes the direction of flight will alter accordingly. This conclusion depends essentially on the hypothesis, made in dealing with narrow planes, that we neglect the shift of the centre of pressure when the angle of attack is varied. In this case if the three forces W, R, H ever pass through the common point G they will always do so, hence the conditions of equilibrium require that the moment about G of the thrust on the rudder vanishes, therefore the thrust itself must vanish and the rudder must always be neutral.

Finally, if the angle of attack is a, the velocity of the aeroplane is determined from the equation

$$R = KSU^2 \sin a$$

or indeed from the assumed law of resistance, supposing the sine law be not assumed to hold good. Thus in conclusion, when there is equilibrium

The inclination of the machine to the horizon depends exclusively on the propeller thrust. Increasing this causes the machine to tip up in front, decreasing it causes the machine to tip down.

The direction of flight is always parallel to the plane of the tail. If this be turned in direction, a corresponding change will take place in the direction of flight.

The magnitude of the velocity U depends on both causes, being determined by the law of resistance.

37. **If the propeller thrust H does not pass through G,** the forces H and R will intersect in a point I, which is fixed relative to the machine. In order for the tail to be neutral the conditions of equilibrium between the forces W, R, H require that GI shall be vertical. If the machine be tilted forwards or backwards this will cease to be the case, and the rudder plane will necessarily cease to be neutral as is evident from the equation of moments. And since the inclination to the horizon depends on the value of H, there will be only one value of H for which the pressure on the tail plane vanishes.

If the engines are stopped and the machine glides steadily under gravity, as it should do with the rudder neutral, then (according to our assumptions) the only forces which maintain equilibrium are R and W, and it follows that R must pass through G. If the engines are re-started and exert a thrust H, the rudder cannot still remain neutral unless the thrust H also passes through G.

Unless, therefore, an aeroplane is provided with substantial fore and aft planes capable of sustaining a considerable pressure, it is important that the propeller thrust should pass as nearly through the centre of gravity as is possible under practical conditions. Otherwise the

machine must be regarded as a doubly-lifting one for all practical purposes. It is very important that in case of a stoppage of the engines the machine should glide "on an even keel" without the rudder being subjected to excessive strains.

A second cause of pressure on the rudder plane is the shifting of the centre of pressure when the angle of attack varies, the main plane being no longer assumed to be "narrow," according to our signification. In this case the rudder, if neutral, remains so, as long as the *angle of attack* remains constant. To secure this, when it is required to rise or descend at an angle θ the propeller thrust must be increased or decreased so as to tilt the machine up or down through the same angle θ. The angle of attack being constant, steering in a vertical plane will have to be effected exclusively by varying the propeller thrust, and not by turning the rudder plane *relative* to the machine.

If, on the other hand, the angle of attack is decreased, the centre of pressure will move forward and the air will have to exert an upward thrust on the rudder plane if it is fixed posteriorly. When the angle of attack is increased, the centre of pressure in general will move backwards and the opposite will be the case.

Double-Lifting Systems.

38. In the general case of a double-lifting aeroplane, where neither the front nor the rear planes are neutral, there are four forces in equilibrium when the machine is moving uniformly. These are the weight W at G, the propeller thrust H and the resultant thrusts R_1 and R_2 on the two surfaces S_1 and S_2 at their centres of pressure A and B. As before we shall suppose *in the first instance* that H passes through the point G. Let the normals to the surfaces at their centres of pressure meet in M,

so that for *narrow* planes M is regarded as a fixed point (Fig. 14). For equilibrium the resultant of R_1 and R_2, which acts through M, must be equal, and opposite to the resultant of W and H acting at G. Calling this resultant R its line of action is along GM, and the planes S_1, S_2 are equivalent to a single plane of suitable area, having its centre of pressure in the line GM and its plane perpendicular to GM. If ω is the inclination to GH of this equivalent plane, the angle between H and R is $90° + \omega$.

As before the three forces at G are represented by the sides of the triangle DEF in Fig. 13, the angle at F being constant and equal to $90° - \omega$, and if H be increased,

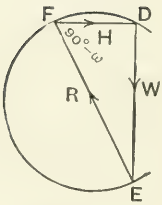

Fig. 13.

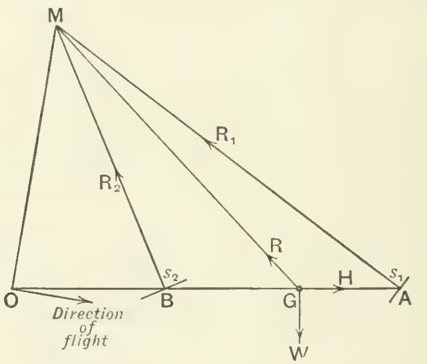

Fig. 14.

it is obvious that F will be lowered relative to D, and the machine will be tilted upwards; conversely, if H be decreased, the machine will be tilted downwards. If the engines are stopped without moving the planes, the aeroplane will be in equilibrium when GM is vertical.

It thus appears that the inclination of the machine to the horizon depends entirely on the propeller thrust. In fact, we may say that, *so far as the equilibrium is concerned, the machine behaves exactly like a glider, the resultant of propeller thrust and gravity taking the place of gravity alone.*

It remains to determine the motion. Knowing R (the

values of H and W being given) the components R_1 and R_2 are known, and if α_1 and α_2 are the angles of attack, we have

$$R_1 = K_1S_1U^2 \sin \alpha_1, \; R_2 = K_2S_2U^2 \sin \alpha_2 \quad . \quad . \quad . \quad (42)$$

Now from the conditions of equilibrium we have

$$R_1 \sin GMA = R_2 \sin GMB$$

Hence

$$K_1S_1 \sin \alpha_1 \sin GMA = K_2S_2 \sin \alpha_2 \sin GMB \quad . \quad . \quad (43)$$

which determines the ratio $\sin \alpha_1 : \sin \alpha_2$. Also $\alpha_1 - \alpha_2$, the angle between the directions of the planes is known, being equal to AMB, and α_1 and α_2 may be found. When this has been done, U^2 may be determined from either of the above equations (42) or otherwise.

39. If one or other of the planes S_1, S_2 is used for steering purposes so that its inclination to the line AB can be varied, it is necessary to have recourse to a geometrical construction in order to ascertain the effects of steering on the position of equilibrium.

Suppose A, G, B *to be in a straight line*, and let MO be drawn through M perpendicular to the direction of flight to meet AB in O. Then $\alpha_1 = AMO$, $\alpha_2 = BMO$, and (43) becomes

$$K_1S_1 \sin AMO \sin AMG = K_2S_2 \sin BMO \sin BMG \quad . \quad (44)$$

But

$$\sin AMO = \frac{AO}{MO} \sin OAM \, ; \; \sin AMG = \frac{AG}{GM} \sin GAM$$

$$\sin BMO = \frac{BO}{MO} \sin OBM \, ; \; \sin BMG = \frac{BG}{GM} \sin GBM$$

whence (44) reduces to

$$K_1S_1AO.AG \sin^2 OAM = K_2S_2BO.BG \sin^2 OBM \quad . \quad (45)$$

again if i_1, i_2 are the angles which the planes S_1, S_2 make with the line AB, $OAM = 90^\circ - i_1$, $OBM = 90^\circ - i_2$, giving

$$K_1S_1AO.AG \cos^2 i_1 = K_2S_2BO.BG \cos^2 i_2 \quad . \quad . \quad (46)$$

This determines the ratio $AO : BO$, and hence O is

found. If i_1, i_2 are small angles, as is generally the case, we may substitute the approximate relation

$$K_1 S_1 AO.AG = K_2 S_2 BO.BG \quad . \quad . \quad . \quad . \quad (46a)$$

based on the approximate assumption that $\cos^2 i_1 = \cos^2 i_2 = 1$. To this degree of approximation the position of O is unaltered by varying the inclinations of the planes to AB.

40. **If the front plane be tilted up** by increasing i_1, then M will be displaced to M' along BM, and the direction of

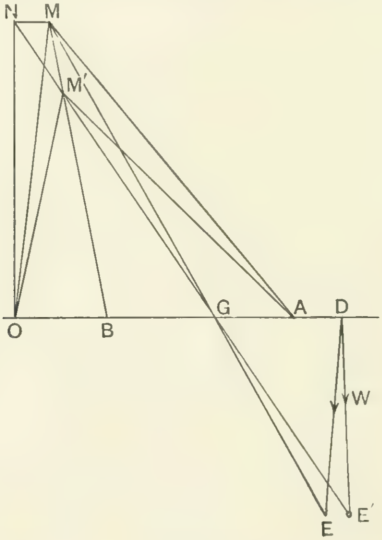

Fig. 15.

flight will be perpendicular to $M'O$ instead of MO (Fig. 15). The effect is to increase the angles of attack, which now become $OM'A$, $OM'B$ (for, by Euclid, these are greater than OMA, OMB respectively), and the direction of R will be changed from MG to $M'G$. The inclination of the machine to the horizon will be altered in consequence, and to estimate the net effect on the direction of flight relative to the horizon, the figure must be combined with the triangle of forces at the point G, say GDE. If, then,

GD is measured containing H units of length, and with centre D and radius W units, we describe a circle cutting MG, $M'G$ in E and E', DE and DE' will be the directions of the vertical relative to the machine. The result of the change will be that the machine will be tilted over forwards through an angle EDE', and since the direction of the line of flight relative to the machine is altered by the amount MOM', *the net result of tilting up the front plane is to depress the direction of the flight path through an angle* $MOM' + EDE'$. In other words, *raising the front plane causes the machine to descend, and conversely depressing the front plane causes it to rise.*

If the machine was originally flying horizontally, OM will be parallel to DE, and if with centre O and radius OM we draw a circle cutting GM' in N, the triangles GOM, GON, will be similar to GDE, GDE', so that ON will be parallel to DE'. In this case *the angle* NOM′ *represents the inclination to the horizon at which the machine begins to descend.*

41. **The effect of tilting up the rear plane** through an angle MBM' is to alter the *relative* direction of the flight path through MOM' (Fig. 16). The angles of attack are altered from OMA, OMB, to $OM'A$ and $OM'B$ respectively. It is obvious that $OM'A < OMA$, so that the angle of attack on S_1 is decreased. That $OM'B < OMB$ is pretty evident from Figure 16. A formal demonstration, however, consists in observing that if OMB, $OM'B$ were equal, the points M, M', O, B would lie on a circle, so that we should have $MM'B = MOB$; but *in the cases which occur in practice* $MM'B$ is small, being the angle between the front and rear planes, whereas MOB is usually somewhere near a right angle, and, at any rate, not a small angle.

The direction of R relative to the machine will be turned through an angle MGM', and the direction of the vertical through EDE' where GDE, GDE' are the two

positions of the force-triangle at G. The net change in the direction of the flight-path relative to the horizon is represented by the difference of angle of $MOM' - EDE'$.

If the machine was originally flying horizontally, and we draw a circle centre O and radius OM cutting GM' in

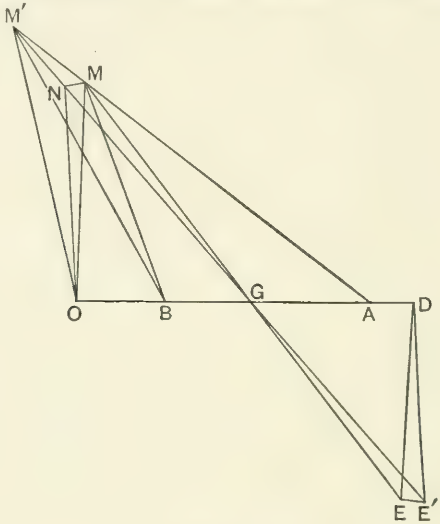

Fig. 16.

N we see that angle $MON = EDE'$ and *the net result of tilting up the rear plane is to cause the machine to rise at an angle* NOM′. *Thus elevating the rear plane causes the machine to rise, and conversely depressing it causes the machine to descend.*

42. **Corrections.**—The above investigation will probably be sufficiently accurate for most practical purposes. It remains, however, to be shown that where the underlying hypotheses are only approximately valid, more accurate results might be obtained by introducing modifications in the construction.

If $\cos^2 i_1$ *and* $\cos^2 i_2$ *are not taken to be unity,* the point O will no longer be fixed when the values of i_1 and i_2 are varied.

If i_1 be increased the ratio $AO : BO$ will be increased,

and O will approach A moving to a position O', and $O'M'$ will cut OM in a point below AB.

The perpendicular to the new flight path relative to the machine will be $M'O'$ instead of MO, and the depression of the path relative to the horizon will be slightly reduced by the angle $OM'O'$.

If on the other hand i_2 be increased $AO:BO$ will be decreased and O will move outwards to a position O', so that the elevation of the line of flight due to the change will be reduced by an amount OMO'. In either case the result is slightly to lessen the steering effect due to turning the plane.

In either case the effect could be more fully studied by constructing the envelope of the line MO, first with i_1 constant and secondly with i_2 constant.

43. **If G does not lie in the line AB**, and we wish to study the effect of varying the inclination of the front plane S_1 relative to the aeroplane, we must join AG

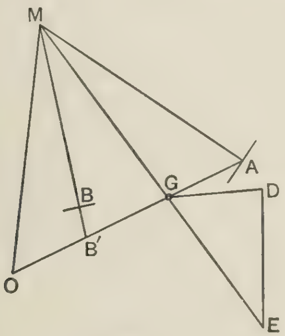

FIG. 17.

meeting BM produced in B', and the point O must be constructed on the line AGB' produced. In fact, since the plane S_2 is kept fixed the construction is just the same as if this plane were shifted parallel to itself to the point B', subject always to the proviso that the inclinations of the planes to the line GA are small (Fig. 17).

If GB be produced to meet OM in O', the point O' will similarly represent the turning point of the line MO when the inclination of S_2 is varied.

44. **Effect of Shifting of Centre of Pressure.**—If, owing to a change in the angles of attack, the centre of pressure of one or both planes is displaced, this will cause a further change in the direction of flight and in the inclination of the aeroplane to the horizon.

In Fig. 18 if AA', BB' represent the displacements

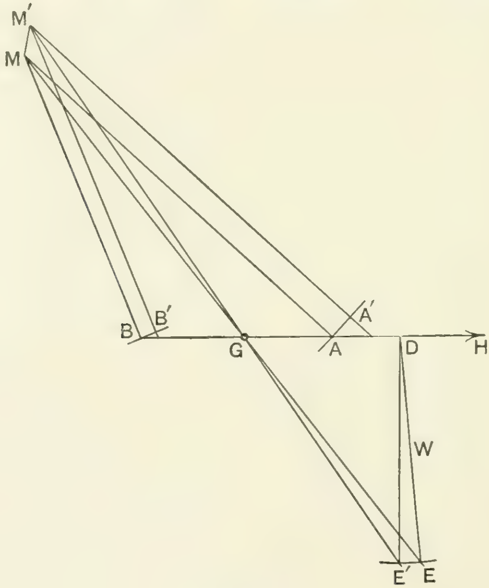

FIG. 18.

of the centres of pressure, MM' will represent the displacement of M, and the constructions of § 39 will give the angle EDE' through which the direction of the aeroplane is turned relative to the vertical. If the centres of pressure are displaced forwards, as occurs when the angles of attack are decreased, the figure shows that the aeroplane will tilt upwards in front through the angle EDE'.

To ascertain the effect on the direction of the flight-

path relative to the aeroplane it is easiest to imagine the angle $A'MB'$ moved backwards and brought into coincidence with AMB, in which case G must be displaced through a distance GG' equal and parallel to $M'M$ (Fig. 19). Here MG' is nearer to MB than MG, and it readily follows from the relation $AO.AG \div BO.BG =$ constant, that O is shifted to a point O' on the side of O remote from B. This depresses the direction of flight relative to the aeroplane through an angle OMO', and as the aeroplane itself has tilted upwards, the net effect is to change the inclination of the flight-path downwards

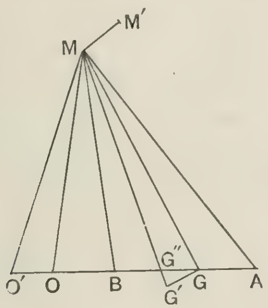

FIG. 19.

through an angle equal to $OMO' - EDE'$, the two causes to a certain extent counterbalancing each other. If GG' be small, and MG' meets AB in G'', it may be shown, by differentiating the relation $AO.AG \div BO.BG =$ constant, that

$$\frac{OO'}{AO.BO} = \frac{GG''}{AG.BG}, \quad . \quad . \quad . \quad . \quad (46b)$$

and, further, that if the aeroplane was initially flying horizontally the effect of the change is to depress or elevate the flight path according as OO' is greater or less than GG''. It is scarcely worth while, however, to reproduce here the discussion of these points in fuller detail.

45. **Prof. Marcel Brillouin's Metacentric Curves.**—A more complete theory of the effects due to displacement of the centre of pressure has been formulated by Prof. Marcel Brillouin.[1] So long as these displacements are neglected, the point M, which is the intersection of the normals to the two planes, may be regarded as the *metacentre* of the machine, being the point through which the resultant thrust, R, always passes. If the planes are broad, and if the line of action of the resultant thrust is constructed for different directions of the relative wind velocity, through the whole range of 360°, these lines of action when marked relative to the planes will envelop a curve, called the *metacentric curve.* It will be noticed that this curve is *not* the locus of the various corresponding positions of M; for the direction of the resultant R depends on the areas S_1, S_2 of the planes, and is thus independent of the direction of the tangent to the locus of M.

M. Brillouin has determined the metacentric curves for certain pairs of planes, assuming Lord Rayleigh's formula for the position of the centre of pressure; and similar methods are obviously applicable, based on Joessel's or any other alternative law. The metacentric curves are in general more or less star-shaped, having eight cusps, four of which correspond to grazing incidence on one or other of the two planes, and the intersection of the normals to the two planes at their geometrical centres is a centre of symmetry of the curve. The existence of four other cusps is most easily shown in the case of two *equal* plane areas.

If the metacentric curve be constructed, and if the propeller-thrust passes through the centre of gravity G, the positions of equilibrium are obtained by drawing tangents to the curve through G, and then determining the direction of the vertical by the force-triangle GDE.

---

[1] *Revue de Mécanique*, 1909.

46. **Case where the propeller-thrust does not pass through the centre of gravity.**—In Fig. 20 the three forces H, W and R will pass through a point Z, which is the intersection of the vertical through G with the line of action of H. Draw GL parallel to the line of action of H, and let it meet the line of action of R in L. Then ZGL is a triangle of forces for the three forces at Z, and we have therefore

$$\frac{H}{GL} = \frac{W}{ZG} = \frac{R}{LZ}.$$

Suppose that W and H are kept constant.

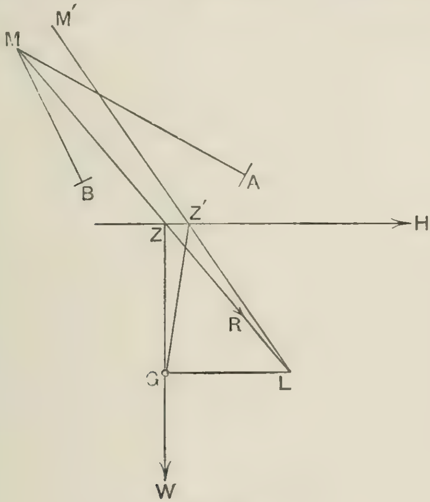

Fig. 20.

If the direction of H *is nearly horizontal*, so that GZ is nearly perpendicular to ZH, then for *small* angular displacements the length GZ is stationary, and L may be regarded as a fixed point through which R always passes. The point L thus takes the place of G in the previous investigations. If the position of M undergoes a small displacement to M' owing to a change in the inclination of one or other of the planes, the new resultant R' will act along LM', cutting ZH in Z', and the new direction of the vertical will be GZ'.

If ZH be not nearly horizontal, the relation $GL : GZ = H : W$ shows that ZL is a tangent to a conic whose focus is G, and directrix is ZH, and the eccentricity (e) of which is equal to the ratio $H : W$.[1] The point of contact P is obtained by making ZGP a right angle, and this is the point about which the line of action of R turns in the case of a small displacement.

Hence, *given the position of the metacentre* M *and the magnitude of* H, *the position of equilibrium is obtained by drawing a tangent from* M *to the conic; and if this cuts* ZH *in* Z, *then* GZ *is the position of the vertical.*

The direction of the flight-normal MO can be constructed by the same method as before, the point of contact P (or L if H is horizontal) taking the place of G in the construction for O.

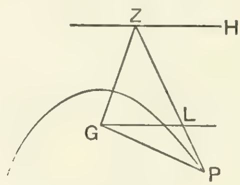

Fig. 21.

Since in an aeroplane H is small compared with W, the conic is an ellipse and two tangents can be drawn to it from the point M, one on either side of G. If, however, we construct the force-triangle for the second tangent, we shall find that it determines a position of equilibrium in which the aeroplane is inverted.

47. **If the engine be stopped**, then, in the new position of equilibrium MG will be the direction of the resultant

[1] The intercept of a tangent to a conic on the latus rectum is e times the distance from the focus of the point at which the tangent meets the directrix. This easily follows from the normal property $SG = eSP$, and the property that PSZ is a right angle, the letter S being substituted for G above and G now being the foot of the normal.

R, and will be vertical. *If G is above the line of action of H* (instead of being below, as in Fig. 21), the effects of the change will be illustrated by Fig. 22. One effect will be to change the relative direction of the vertical from GZ to GZ' (in MG produced) thus causing the aeroplane to dip downwards in front through an angle ZGZ'. But as R moves forward from MZ to MZ', the flight normal will also move forward from MO to a new position MO', so that the angles of attack of both planes will be decreased by an amount OMO'. The net result will be to depress the direction of the flight-path relative

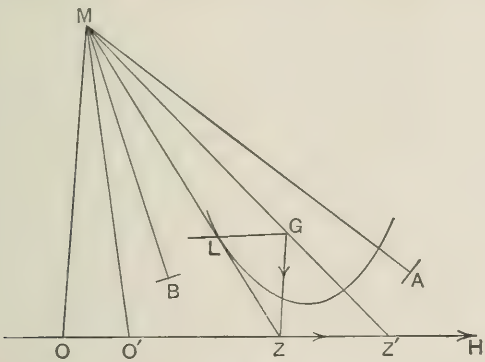

FIG. 22.

to the horizon by an amount $ZGZ' - OMO'$. If instead of stopping the engine the propeller-thrust H be reduced, the eccentricity of the conic to which MZ is a tangent will be reduced in the same proportion and in Fig. 22 MZ will therefore be displaced towards MGZ', so that the resulting changes in the position of equilibrium will be of the same kind as before, but less in amount. For an increase in the propeller-thrust the effects will be reversed, and the aeroplane will begin to rise.

If it were possible to make the angles ZGZ' and OMO' equal, the direction of the flight-path would be unaffected by stopping the engines, and under closely

analogous circumstances the direction of the flight-path might be made independent of the propeller-thrust. This arrangement would be advantageous, but it is not in general practicable, as Brillouin has shown in the paper above referred to. If, as is usually the case, the angles of attack are small, MG will be large compared with the dimensions of the aeroplane, and it follows that the angle ZGZ' is large compared with ZMZ' unless GZ', and therefore the distance of the propeller-axis below G, are also large compared with the other dimensions of the aeroplane, a condition impracticable for other fairly obvious reasons. The only possible alternative would be to make OMO' large compared with ZMZ'. Even if this were mathematically possible (which appears not to be the case), it would have the result that a very small displacement of the centre of gravity would produce a very large change in the direction of the flight-path, a condition possessing obvious grave objections.

The reader will have no difficulty in drawing the corresponding figure, with the aid of Figs. 20 and 21, for the case where the propeller-axis is below the centre of gravity. In this case, when the engine is stopped, or the propeller-thrust is reduced, the angular displacements of the lines GZ and MO will be in opposite senses so that if GZ' and MO' denote their new positions the direction of the flight-path will be depressed through an angle $ZGZ'+OMO'$. At the same time the angles of attack on both planes will be increased by OMO', so that the descent will be less violent than it would otherwise be. Conversely, if the propeller-thrust be increased, the direction of the flight-path will be correspondingly elevated.

CHAPTER V

LONGITUDINAL STABILITY OF SINGLE-LIFTING SYSTEMS

The Simplest Case.

Single Lifting Plane Propelled Horizontally by a Central Thrust—Lanchester's Condition.

48. In order to avoid algebraic complications at the outset, we first consider a system specified as follows :—

Two surfaces (either of which may be replaced by a

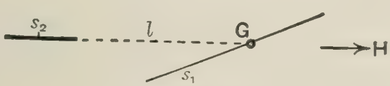

Fig. 23.

pair of superposed surfaces), S_1, S_2, of which the front surface S_1 supports the whole weight of the aeroplane, being inclined to the line of flight at an angle of attack α, while the rear surface S_2 acts as a tail or rudder or auxiliary plane, being placed in a neutral direction so that $\alpha_2 = 0$. Distance between centres of pressure $= l$.

The line of action of the propeller-thrust passes through the centre of gravity (and this fact will in the future be represented by the statement that the *thrust is central*), the direction of the thrust being along the line of flight, which in this section is taken to be horizontal. In these circumstances the resultant thrust of the air on S_1 also

passes through the centre of gravity; thus the centre of gravity either coincides with the centre of pressure or is on the normal to S_1 through the centre of pressure. In the former case we have for S_1 both ξ' and $\xi''=0$, $x=0$ and $y=0$; in the latter $\xi'=0$ exactly and $\xi''=0$ approximately, while for S_2 we have $\xi'=\xi''=-l$.

The conditions of equilibrium give:

$$
\begin{aligned}
H &= KS_1U^2 \sin^2 a \\
W &= KS_1U^2 \sin a \cos a \\
0 &= KS_1U^2\xi' \sin a, \text{ or } \xi' = 0 \text{ as above} \quad . \quad . \quad . \qquad (47)
\end{aligned}
$$

The nine derivatives are

$$
\begin{array}{lll}
X_u = 2KS_1U\sin^2 a, & X_v = KS_1U\sin a\cos a, & X_r = 0 \\
Y_u = 2KS_1U\sin a\cos a, & Y_v = KS_1U\cos^2 a + KS_2U, & Y_r = -KS_2Ul \\
N_u = 0, & N_v = -KS_2Ul, & N_r = KS_2Ul^2 \quad (48)
\end{array}
$$

We obtain

$$\mathfrak{A}_o = CW^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49a)$$

$$\frac{\mathfrak{B}_o}{gU} = CWK\{S_1(1+\sin^2 a) + S_2\} + W^2KS_2l^2 \quad . \quad . \quad . \qquad (49b)$$

$$\frac{\mathfrak{C}_o}{g^2U^2} = 2CK^2S_1S_2\sin^2 a + WK^2S_1S_2l^2(1+\sin^2 a) + \frac{W^2}{g}KS_2l \qquad (49c)$$

$$\frac{\mathfrak{D}_o}{g^3U^3} = \frac{2W}{g}K^2S_1S_2l\sin^2 a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49d)$$

$$\frac{\mathfrak{E}_o}{g^4U^4} = \frac{2W}{U^2g}K^2S_1S_2l\sin a\cos a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (49e_1)$$

It will be noticed that the conditions $\mathfrak{D}_o$ and $\mathfrak{E}_o$ positive require l to be positive, that is, *the auxiliary plane must be placed behind the main plane*, as a tail, not in front.

In virtue of the condition of equilibrium we have also

$$
\begin{aligned}
\frac{\mathfrak{E}_o}{g^4U^4} &= \frac{2K^3}{g}S_1{}^2S_2l\sin^2 a\cos^2 a \\
&= \frac{2K^3}{g}S_1{}^2S_2l\sin^2 a\,(1-\sin^2 a) \quad . \quad . \qquad (49e_2)
\end{aligned}
$$

When expressed in this form we see that $\mathfrak{D}_o/U^3$ and $\mathfrak{E}_o/U^4$ contain the factor $\sin^2 \boldsymbol{a}$, while the three previous coefficients remain finite in the limit when $\boldsymbol{a}=0$. In the biquadratic for λU, the last two coefficients are therefore small in comparison with the first three, and approximate

methods of solution may be used. In particular we notice that in the expression $\mathfrak{B}_o\mathfrak{C}_o\mathfrak{D}_o - \mathfrak{C}_o\mathfrak{B}_o^2 - \mathfrak{A}_o\mathfrak{D}_o^2$, the first two terms contain $\sin^2 a$ as a factor, while $\mathfrak{A}_o\mathfrak{D}_o^2$ contains $\sin^4 a$; the latter may therefore be neglected in a first approximation and the condition of stability then reduces on division by $\mathfrak{B}_o$ to

$$\mathfrak{C}_o\mathfrak{D}_o - \mathfrak{E}_o\mathfrak{B}_o > 0 \quad . \quad . \quad . \quad . \quad . \quad (50)$$

Now neglecting $\sin^2 a$ we have

$$\frac{\mathfrak{B}_o}{gU} = KW\{(S_1 + S_2)C + Wl^2S_2\} \quad . \quad . \quad . \quad (51b)$$

$$\frac{\mathfrak{C}_o}{g^2U^2} = KWS_2l\{KS_1l + \frac{W}{g}\} \quad . \quad . \quad . \quad . \quad (51c)$$

and using the first form of $\mathfrak{E}_o/g^4U^4$ we obtain from (50) the approximate *primitive condition of stability*

$$\left\{\frac{W}{g} + \underline{KS_1l}\right\}S_2U^2l \tan a - \{(S_1 + S_2)C + \underline{Wl^2S_2}\} > 0 \quad . \quad (52)$$

Making use, however, of the equations of equilibrium (47) we have

$$\begin{aligned} Wl^2S_2 &= KS_1S_2l^2U^2 \sin a \cos a \\ &= KS_1S_2l^2U^2 \tan a\,(1 - \sin^2 a) \\ &= KS_1S_2l^2U^2 \tan a \end{aligned}$$

to a first approximation. Hence the terms underlined cancel to this order of approximation, leaving

$$\frac{W}{g}S_2U^2l \tan a - (S_1 + S_2)C > 0 \quad . \quad . \quad . \quad (53)$$

whence if $C = Wk^2$, so that k is the radius of gyration about the axis of z,

$$\frac{U^2}{g} > \frac{S_1 + S_2}{S_2} \cdot \frac{k^2}{l \tan a} \quad . \quad . \quad . \quad . \quad (54)$$

This is the condition of stability obtained by Lanchester by an entirely independent method. As use has been made of the conditions of equilibrium in cancelling the parts underlined in (52) we ought to regard it as a modified form (to this extent) of the conditions of stability.

Another *modified condition of stability* can be obtained by expressing U^2 in terms of W by means of the condition of equilibrium, again neglecting the difference between $\tan\alpha$ and $\sin\alpha\cos\alpha$. We thus obtain

$$\frac{l}{g} > \frac{Kk^2}{W} \frac{(S_1 + S_2)S_1}{S_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (54a)$$

This form is probably better than the previous one, as it only involves quantities depending on the weight and measurements of the aeroplane, in addition to the coefficient of resistance K. If the propeller-thrust, velocity of propulsion U and angle of attack α are varied in a way consistent with the conditions of equilibrium the stability determined by ($54a$) is independent of their variations.

49. If it is desired to proceed to a higher degree of approximation in which $\cos\alpha$ is not taken to be equal to unity, the best plan is to write, as in § 34, $S'_1 = S_1\cos^2\alpha$ and $\mu = \tan\alpha$ in the equations of equilibrium and the coefficients, and we thus obtain,

$$H = KS'_1U^2\mu^2 \quad W = KS'_1U^2\mu \quad . \quad . \quad . \quad . \quad (55)$$

$$\mathfrak{A}_0 = W^3k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56a)$$

$$\frac{\mathfrak{B}_0}{gU} = W^2K\{k^2(S'_1 + S_2) + l^2S_2 + 2k^2S'_1\mu^2\} \quad . \quad . \quad . \quad (56b)$$

$$\frac{\mathfrak{C}_0}{g^2U^2} = WKS_2\{KS'_1l^2 + \frac{W}{g}l + 2KS'_1(l^2 + k^2)\mu^2\} \quad . \quad . \quad (56c)$$

$$\frac{\mathfrak{D}_0}{g^3U^3} = \frac{2W}{g}K^2S'_1S_2l\mu^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56d)$$

$$\frac{\mathfrak{E}_0}{g^4U^4} = \frac{2W}{U^2g}K^2S'_1S_2l\mu = \frac{2K^3}{g}S'^2_1S_2l\mu^2 \quad . \quad . \quad . \quad . \quad . \quad (56e)$$

$$\frac{\mathfrak{B}_0\mathfrak{C}_0\mathfrak{D}_0 - \mathfrak{E}_0\mathfrak{B}_0^2 - \mathfrak{A}_0\mathfrak{D}_0^2}{g^2U^2WK\mathfrak{B}_0\mathfrak{D}_0} = \frac{WS_2l}{g} - Kk^2S'_1(S'_1 + S_2)$$
$$+ 2KS'_1\mu^2\{S_2(l^2 + k^2) - S'_1k^2\} - 2\frac{W}{g}\frac{S'_1S_2k^2l\mu^2}{S_2(l^2 + k^2) + S'_1k^2(1 + 2\mu^2)}. \quad (57)$$

$$= \frac{WS_2l}{g} - Kk^2S_1\cos^2\alpha(S_1\cos^2\alpha + S_2) + 2KS_1\sin^2\alpha\{S_2(l^2 + k^2) - S_1k^2\cos^2\alpha\}$$
$$- 2\frac{W}{g}\frac{S_1S_2k^2l\sin^2\alpha}{S_2(l^2 + k^2) + S_1k^2(1 + \sin^2\alpha)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57a)$$

and the condition for stability requires that this expres-

sion shall be positive. Putting $\cos a = 1$, $\sin a = 0$, this reduces to the condition previously obtained.

The foregoing expressions give the *exact* conditions of stability whether the angle a be large or small, provided that the resistance follows the sine law, and that the shift of the centre of pressure is negligible when the angle of attack is varied. In view, however, of these assumptions not being exactly fulfilled in actual practice it is probably better to be satisfied with the approximate solutions obtained by neglecting $\sin^2 a$, leading to

$$\frac{W}{KS_1 \cos^2 a} \text{ or } U^2 \tan a > \frac{k^2 g}{l} \frac{S_1 \cos^2 a + S_2}{S_2}$$

or the even simpler forms (54), (54a) obtained by substituting unity for $\cos^2 a$ in these. In the S'_1, μ notation, the above conditions may be written

$$\frac{W}{KS'_1} \text{ or } U^2\mu > \frac{k^2 g}{l} \frac{S'_1 + S_2}{S_2} \quad . \quad . \quad . \quad (58)$$

and these we shall regard as the *standard forms* of the condition of stability as it will appear later on in our work, that the "S'_1, μ notation" considerably simplifies the algebra.

Separation of the long and short oscillations.

50. In the biquadratic equation for λ, the last two coefficients (49d, e_2) contain the factor $\sin^2 a$, so that if a is small these coefficients are small quantities of the second order in comparison with the others. This fact enables us to obtain approximate solutions of the biquadratic, and to separate the different oscillations of the system.

Let us write the biquadratic for λ

$$\mathfrak{A}\lambda^4 + \mathfrak{B}\lambda^3 + \mathfrak{C}\lambda^2 + \mathfrak{D}''\lambda + \mathfrak{E}'' = 0,$$

the suffixes being dropped as they were only used to distinguish the coefficients for symmetric and asymmetric

oscillations, and the accents in $\mathfrak{D}''$ and $\mathfrak{E}''$ being used to show that these are small quantities of the second order.

If λ is finite, then for a first approximation we may neglect $\mathfrak{D}''$ and $\mathfrak{E}''$, and the equation on dividing through by λ^2 becomes

$$\mathfrak{A}\lambda^2 + \mathfrak{B}\lambda + \mathfrak{C} = 0$$

The roots of this equation determine oscillations whose times remain finite, however small be the angles of attack, and we shall call these the *short oscillations*. If, on the other hand, λ is small, $\mathfrak{D}''\lambda$ may be neglected in comparison with $\mathfrak{E}''$ and $\mathfrak{A}\lambda^4$, and $\mathfrak{B}\lambda^3$ in comparison with $\mathfrak{C}\lambda^2$. Hence to the lowest order of approximation we get

$$\mathfrak{C}\lambda^2 + \mathfrak{E}'' = 0$$

or

$$\lambda = \pm i \sqrt{\frac{\mathfrak{E}''}{\mathfrak{C}}} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

This solution is of the first order of small quantities, and the values of λ being imaginary the motion is oscillatory. To find a closer approximation we substitute the value of λ^2 given above in $\mathfrak{B}\lambda^3$ (still neglecting $\mathfrak{A}\lambda^4$) and write the equation

$$\mathfrak{C}\lambda^2 + \left\{\mathfrak{D}'' - \frac{\mathfrak{E}''\mathfrak{B}}{\mathfrak{C}}\right\}\lambda + \mathfrak{E}'' = 0 \quad . \quad . \quad . \quad (60)$$

showing that to this order the real part of the values of λ is equal to

$$\frac{\mathfrak{E}''\mathfrak{B} - \mathfrak{C}\mathfrak{D}''}{2\mathfrak{C}^2}$$

The small values of λ thus determine *slow or long oscillations* whose period approximates to $2\pi\sqrt{(\mathfrak{C}/\mathfrak{E}'')}$ and modulus of decay is equal to

$$\frac{\mathfrak{C}\mathfrak{D}'' - \mathfrak{E}''\mathfrak{B}}{2\mathfrak{C}^2}$$

and is positive if the approximate condition for stability is satisfied, namely, $\mathfrak{C}\mathfrak{D}''-\mathfrak{E}''\mathfrak{B} < 0$.

To examine the character of the oscillations we must go back to the equations of motion, which are

$$\left(2S_1 \sin^2 a + \frac{W\lambda}{KUg}\right)u + S_1 \sin a \cos a \,.\, v \qquad - \frac{W}{KU\lambda}r \qquad = 0 \quad (61u)$$

$$2S_1 \sin a \cos a\, u + \left(S_1 \cos^2 a + S_2 + \frac{W\lambda}{KUg}\right)v + \left(-S_2 l + \frac{W}{Kg}\right)r = 0 \quad (61v)$$

$$0u \qquad + \qquad (-S_2 l)v \qquad + \left(S_2 l^2 + \frac{Wk^2\lambda}{KUg}\right)r = 0 \quad (61r)$$

51. Short oscillations.—If we neglect *sin a* altogether the equations become

$$\frac{\lambda^2}{g}u - r = 0 \quad . \quad . \quad (62u)$$

$$\left(S_1 + S_2 + \frac{W\lambda}{KUg}\right)v + \left(-S_2 l + \frac{W}{Kg}\right)r = 0 \quad . \quad . \quad (62v)$$

$$-S_2 l v \qquad + \left(S_2 l^2 + \frac{Wk^2\lambda}{KUg}\right)r = 0 \quad . \quad . \quad (62r)$$

and these are applicable to the *short oscillations.* The equation for λ is in this case obtained by eliminating v and r from the last two equations, and this gives

$$k^2\frac{W^2\lambda^2}{K^2U^2g^2} + \frac{W\lambda}{KUg}\{k^2(S_1 + S_2) + l^2S_2\} + S_1S_2l^2 + S_2l\frac{W}{Kg} = 0 \quad . \quad (63)$$

This equation is identical with the equation $\mathfrak{A}\lambda^2 + \mathfrak{B}\lambda + \mathfrak{C} = 0$ if we take the values of $\mathfrak{A}_0$, $\mathfrak{B}_0$, $\mathfrak{C}_0$ given in § 48, and it is thus seen that the motions here considered are really the short oscillations of the system.

The ratio $v : r$ is given by either of these equations, say the second, which gives

$$\frac{v}{r} = l + \frac{Wk^2\lambda}{KUgS_2l}$$

or, if preferred,

$$\frac{v - lr}{r} = \frac{Wk^2\lambda}{KUgS_2l} \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where $v - lr$ is obviously the velocity of the tail plane perpendicular to the axis of x. This equation is of course merely the equation of moments, being equivalent to

$$Wk^2\frac{dr}{gdt} = KUS_2(v - lr)l.$$

The velocity u is found in terms of r from the equation

$\lambda^2 u/g + r = 0$, which is derived from the equation of motion

$$\frac{W}{g}\frac{du}{dt} = -W\sin\theta \text{ where } \frac{d\theta}{dt} = r \qquad (65)$$

the assumption being in this case that *the effects of the air pressures on the planes may be neglected in considering the motion along the line of flight.*

Since, however, u does not enter into the two equations from which the quadratic for λ is found, it follows that *the short oscillations are independent of the variations of velocity in the line of flight and are determined entirely by the normal and rotational motions,* v *and* r.

In the next place we observe that λ only enters into the equation (63) in the form λU, and that the values of λU given by this equation are independent of the velocity U. The rates of oscillation are thus proportional to the velocity U. If, instead of taking the time as independent variable, we take the distance flown, s, where $s = Ut$, we see that *the actual path described by the machine when oscillating is independent of* U, *the velocity of flight,*[1] so that if the velocity of flight is changed the machine will still perform the same number of oscillations, and these oscillations will subside to the same fraction of their initial amplitude when the machine travels over the same distance.

The condition of stability requires that in the equation for λ or λU, all the coefficients shall be positive and therefore

$$k^2(S_1 + S_2) + l^2 S_2 > 0 \qquad (66a)$$

$$S_1 S_2 l^2 + S_2 l \frac{W}{Kg} > 0 \qquad (66b)$$

The first expression is essentially positive. The second will be positive if l is positive, *or* if l is negative and numerically greater than W/KS_1g.

In the former case *the auxiliary plane is behind the*

[1] Assuming, of course, that u is neglected.

main planes. The corresponding condition in the case of a doubly lifting machine is that *the front plane must have a larger angle of attack than the rear one*, as will be shown later.

The negative solution is interesting in connection with aeroplanes having the rudder planes in front (*e.g.* the Wright machine). The conclusion is that *by placing the rudder either behind or sufficiently far forward in front of the main planes it is possible to secure stability so far as the short oscillations are concerned.*

The motion will be either really oscillatory or subsiding in character, according to whether the values of λU are

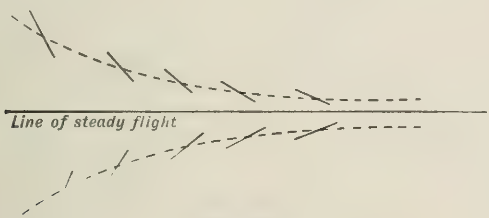

Fig. 24.

complex or real; the conditions being $\mathfrak{B}^2 - 4\mathfrak{A}\mathfrak{C}$ negative or positive, respectively, that is

$$\{k^2(S_1 + S_2) + l^2S_2\}^2 - 4k^2\left(S_1S_2l^2 + S_2l\frac{W}{Kg}\right) < \text{ or } > 0$$

This condition may be transformed into

$$\{k^2(S_1 - S_2) - l^2S_2\}^2 + 4k^4S_1S_2 - 4k^2S_2l\frac{W}{Kg} < \text{ or } > 0 \quad . \quad (67)$$

and it will thus be seen that if l is negative, or if l is positive and less than a certain limit, the disturbance is subsiding in character instead of being periodic.

Taking now, as an example, the case where λ is real and negative so that l is positive, equation (64) shows that $v - lr$ and r are of opposite signs, and hence the angular displacement of the aeroplane is in the opposite direction to the displacement of the tail plane. The motion of the tail plane is represented diagrammatically in Fig. 24 for

two different displacements, one above and the other below the line of flight. Other cases may be similarly discussed.

Long Oscillations.

52. For the second solution of the biquadratic, the values of λ are to the lowest sufficient order of approximation

$$\lambda = \pm i \sqrt{\frac{\mathfrak{E}}{\mathfrak{C}}} - \frac{\mathfrak{C}\mathfrak{D} - \mathfrak{E}\mathfrak{B}}{2\mathfrak{C}^2} \quad . \quad . \quad . \quad (68)$$

the imaginary part being of the first order in a and the real part, determining the modulus of decay, being of the second order. To obtain a general idea of the character of the oscillations we shall neglect the latter part, and we thus assume the oscillations to be periodic.

Writing S'_1 for $S_1 \cos^2 a$ and μ for $\tan a$, equations (61) give

$$\left(2S'_1\mu^2 + \frac{W\lambda}{KUg}\right)u + S'_1\mu v \quad - \quad \frac{W}{KU\lambda}r = 0 \quad . \quad . \quad (69u)$$

$$2S'_1\mu\, u + \left(S'_1 + S_2 + \frac{W\lambda}{KUg}\right)v + \left(-S_2 l + \frac{W}{Kg}\right)r = 0 \quad . \quad . \quad (69v)$$

$$-S_2 lv \quad + \left(S_2 l^2 + \frac{Wk^2\lambda}{KUg}\right)r = 0 \quad . \quad . \quad (69r)$$

With μ small, the only way of obtaining a solution differing essentially from that obtained for the short oscillations is by assuming that u is large compared with v and r so that $u \sin a$ or $u\,\mu$ is comparable with v and r. Moreover, in view of the fact that the terms containing λ in the second and third equations (61) were previously comparable with the others, it follows that for the long oscillations these terms may be neglected for a first approximation.

If we neglect λ in the third equation we obtain

$$-v + lr = 0 \quad . \quad . \quad . \quad . \quad . \quad (70)$$

or *the normal velocity of the tail is zero.* More generally the equation, which may be written

$$v - lr = \frac{Wk^2\lambda}{KS_2Ulg}r \quad . \quad . \quad . \quad . \quad . \quad . \quad (70a)$$

shows that the normal velocity of the tail is in every case small.

Again we may eliminate S'_1 and S_2 from the three equations by multiplying the first by $-1/\mu$, the second by 1, and the third by $1/l$ and adding, and we get on removing superfluous factors

$$-\frac{\lambda u}{\underline{\mu}} + \lambda v + \left(\frac{g}{\underline{\lambda\mu}} + U + \frac{k^2\lambda}{l}\right) r = 0 \quad . \quad . \quad . \quad (71)$$

and this, since it does not involve S_1 and S_2, must be independent of the resistances and must represent the *equation of moments about the metacentre.* Observing that u is assumed large compared with v, the terms of lowest order are those underlined, leading to

$$u\lambda^2 = gr \quad . \quad . \quad . \quad . \quad . \quad . \quad (71a)$$

On the other hand, if we neglect λ in the second and third equations, we have, on eliminating S_2,

$$2S'_1\mu u + S'_1 v + \frac{W}{Kg}v = 0 \quad . \quad . \quad . \quad . \quad (72)$$

or substituting $v = lr$ again, we obtain

$$2S'_1\mu u + \left(S'_1 l + \frac{W}{Kg}\right) r = 0 \quad . \quad . \quad . \quad . \quad (73)$$

From (73) and (71a):

$$\lambda^2 = -\frac{2\mu g}{l + \dfrac{W}{gKS'_1}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

which agrees with the first approximation for the long oscillations given by

$$\lambda^2 = -\frac{\mathfrak{E}_0}{\mathfrak{C}_0} = -g^2\frac{2WK^2S'_1S_2l\mu}{g\{WK^2S'_1S_2l^2 + W^2KS_2l/g\}}$$

taking the values of the coefficients in § 49, and neglecting terms containing μ^2 in the value of $\mathfrak{C}_0$.

53. To trace the undulating path described by the centre of gravity, we suppose that the co-ordinates of this point at a time t are $\xi + Ut$ and η, the axis of x

coinciding with the direction of steady motion. Then if θ be the angular co-ordinate

$$\frac{d\xi}{dt} + U = (U + u)\cos\theta - v\sin\theta \quad . \quad . \quad . \quad (75a)$$
$$= U + u \text{ to a first approximation}$$
$$\frac{d\eta}{dt} = (U + u)\sin\theta + v\cos\theta$$
$$= U\theta + v \text{ to a first approximation} \quad . \quad (75b)$$

and since ξ, η, θ all vary as $e^{\lambda t}$,

$$\lambda\xi = u \text{ or } \xi = \frac{u}{\lambda}$$

$$\lambda\eta = U\theta + v = U\frac{v}{\lambda} + v \text{ or } \eta = U\frac{v}{\lambda^2} + \frac{v}{\lambda}$$

Assuming the approximate expressions $v = \lambda^2 u/g$, $v = lr$, we have finally

$$\eta = \frac{\xi}{g}(U\lambda + l\lambda^2)$$

or writing $\lambda = \lambda' i$ where $i = \sqrt{-1}$,

$$\eta = \frac{\xi}{g}(U\lambda' i - l\lambda'^2)$$

where λ' is real.

Put $\xi = A(\cos\lambda' t + i\sin\lambda' t)$ and equate the real parts, then we get the solution

$$\left.\begin{aligned} \xi &= A\cos\lambda' t \\ \eta &= -\frac{A}{g}\{U\lambda'\sin\lambda' t + l\lambda'^2\cos\lambda' t\} \end{aligned}\right\} \quad . \quad (76)$$

If we write $x = \xi + Ut$, $y = \eta$, then x, y will be the co-ordinates of the centre of gravity referred to a *fixed* origin, so that the equations of the actual path will be given by

$$\left.\begin{aligned} x &= Ut + A\cos\lambda' t \\ y &= -\frac{A}{g}\{U\lambda'\sin\lambda' t + l\lambda'^2\cos\lambda' t\} \end{aligned}\right\} \quad . \quad (77)$$

and to trace the path it is only necessary to add the ordinates of the two curves determined, one by the equations

$$x = Ut + A\cos\lambda' t \quad y = -\frac{AU}{g}\lambda'\sin\lambda' t \quad . \quad . \quad (78)$$

and the other by the equations

$$x = Ut + A\cos\lambda' t \qquad y = -\frac{Al\lambda'^2}{g}\cos\lambda' t \qquad . \quad . \qquad (79)$$

of which the first represents waves having the crests more pointed than the troughs (upper curve, Fig. 26), while the second represents waves the descending parts of which are steeper than the ascending ones (lower curve). The actual path, which is a combination of the two, should be familiar to anyone who has thrown paper gliders.

In regard to the assumption $v=\lambda^2 u/g$, in connection with the fact that we have retained terms containing λ'^2 in the expression for η, it should be noticed that the terms

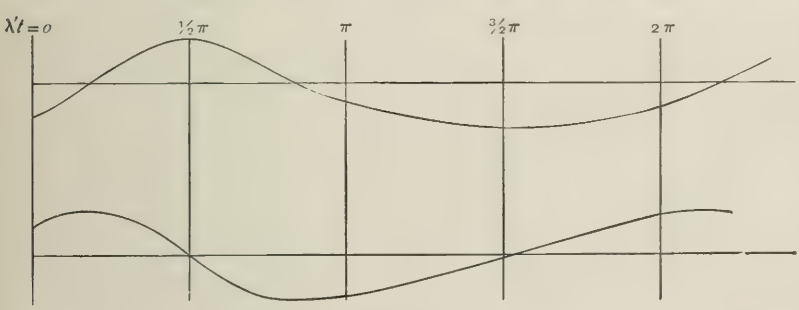

Fig. 25.

neglected in (71a) would only affect the final result by quantities *of a higher order than the second.*

The *path of the centre of pressure of the tail plane* is very approximately the first curve of Fig. 25, the equation of motion being

$$\lambda\eta = U\theta + (v - lr)$$

where $v-lr$ is a negligible quantity.

Effect of inclination of propeller axis.

54. If the axis of the propeller is not along the direction of motion, the primitive conditions of stability will,

of course, be unchanged, but as the conditions of equilibrium now give

$$W + H \sin \eta = KS'_1 U^2 \mu \quad . \quad . \quad . \quad . \quad (80)$$

the terms previously cancelled in modifying the conditions of stability will no longer cancel. The approximate condition of stability corresponding to (54)

$$\frac{U^2}{g} > \left\{ \frac{S'_1 + S_2}{S_2} \frac{k^2}{l} - \frac{H \sin \eta}{W} l \right\} \cot a \quad . \quad . \quad (81)$$

while corresponding to (54a) we shall have

$$\frac{l}{g} > \frac{KS'_1}{W + H \sin \eta} \left\{ \frac{S'_1 + S_2}{S_2} k^2 - \frac{H \sin \eta}{W} l^2 \right\} \quad . \quad (81a)$$

According to (81a), if η is positive so that *the propeller thrust tends to depress the machine, stability is increased, and the reverse is the case if the propeller thrust tends to elevate the machine.*

Effect of inclination of the flight path.

55. That the stability, both symmetric and asymmetric, of an aeroplane depends very largely on the angle which the direction of flight makes with the horizon was first pointed out to me by Mr. E. H. Harper, to whom the following investigations are largely due. In this section and the next we assume the propeller thrust to be central and acting along the direction of motion, so that the tail plane is neutral and the shift of the centre of pressure is neglected.

The first three coefficients of the biquadratic are as given in (49a—e), but we have

$$\frac{\mathfrak{D}_o}{g^3 U^3} = \frac{2W}{g} K^2 S_1 S_2 l \sin^2 a + \frac{W^2}{gU^2} KS_2 l \sin \theta \quad . \quad . \quad (82d)$$

$$\frac{\mathfrak{E}_o}{g^4 U^4} = \frac{2W}{U^2 g} K^2 S_1 S_2 l \{\sin a \cos a \cos \theta + \sin^2 a \sin \theta\} \quad . \quad (82e)$$

In virtue of the equation of equilibrium

$$W \cos \theta = KS_1 U^2 \sin a \cos a \quad . \quad . \quad . \quad (83)$$

these reduce to

$$\frac{\mathfrak{D}_o}{g^3U^3} = \frac{W}{g} K^2S_1S_2l(2\sin^2 a + \sin a \cos a \tan\theta)$$

$$= \frac{W}{g} K^2S_1S_2l \sin a \cos a\,(2\tan a + \tan\theta) \quad . \quad . \quad (84d)$$

$$\frac{\mathfrak{E}_o}{g^4U^4} = \frac{2W}{U^2g} K^2S_1S_2l \sin a \cos(\theta - a) \quad . \quad . \quad . \quad . \quad (84e)$$

Hence $\mathfrak{D}_o$ *vanishes when the machine is ascending at an angle given by* $\tan\theta = -2\tan a$. In this case the discriminant $\mathfrak{H}_o$ becomes equal to $-\mathfrak{E}_o\mathfrak{B}_o^2$, and is negative, so that the flight is distinctly unstable. The limit for the angle of elevation of the flight path at which stability ceases is thus *less* than the value given by $\tan\theta = -2\tan a$, showing that, even other things being the same, *a machine rapidly loses its stability when it begins to rise in the air.*

It should be noticed that $\mathfrak{E}_o$ vanishes when $\theta - a = 90°$, but this condition is not likely to occur in practice.

On the other hand, a positive value of θ is favourable for stability, so that *if an aeroplane is symmetrically stable when flying horizontally it will be more stable when descending.*

An important consequence is that *it is not safe to draw inferences regarding the stability of motor-driven machines from experiments with gliders.*

56. For purposes of approximation and simplification we shall suppose θ and a both small so that $\cos(\theta - a)$ may be taken to be equal to unity; in such cases the value of $\mathfrak{D}_o$ alone is affected by the inclination of the flight path. Corresponding to the approximate condition (52) we now have

$$\left\{\frac{W}{g} + K_1S'_1l\right\}S_2U^2l(\tan a + \tfrac{1}{2}\tan\theta) - \{(S'_1 + S_2)C + Wl^2S_2\} > 0 \quad (85)$$

Making use of the conditions of equilibrium, we may write the modified condition in the form corresponding to (53)

$$\frac{U^2}{g} > \frac{(S'_1 + S_2)k^2 - S_2l^2\tan\theta/2\tan a}{S_2l(\tan a + \tfrac{1}{2}\tan\theta)} \quad . \quad (86)$$

this being the modification required to adapt Lanchester's condition to flight at an inclination θ to the horizon. For the particular case where the machine is descending as a glider, so that $\theta = a$, this condition gives

$$\frac{U^2}{g} > \frac{2}{3} \frac{(S'_1 + S_2)k^2 - \frac{1}{2}S_2l^2}{S_2 l \tan a} \quad . \quad . \quad . \quad (87)$$

so that the minimum velocity-height ($U^2/2g$) required for stability is less than two-thirds that required by Lanchester's condition.

On the other hand, if the machine is ascending at an angle equal to the angle of attack, so that $\theta = -a$, the condition becomes

$$\frac{U^2}{g} > 2 \frac{(S'_1 + S_2)k^2 + \frac{1}{2}S_2l^2}{S_2 l \tan a} \quad . \quad . \quad . \quad (88)$$

requiring a value of U^2/g, more than double that required for stability in horizontal flight.

From the fact that $\mathfrak{A}_o$, $\mathfrak{B}_o$, $\mathfrak{C}_o$ are independent of θ, and $\mathfrak{E}_o$ only contains θ in the form $\cos(\theta - a)$, which is nearly equal to unity, it follows that *to our assumed degree of approximation, the short oscillations are independent of the inclination of the flight path, as is also the period of the long oscillations; on the contrary, the modulus of decay of the latter depends greatly on the inclination.*

Effect (*a*) of head resistances and (*b*) of variable propeller thrust.

⁂ In what follows we shall omit the suffixes in $\mathfrak{A}_o$ $\mathfrak{B}_o$ $\mathfrak{C}_o$ $\mathfrak{D}_o$ $\mathfrak{E}_o$, the context showing that we are dealing with longitudinal stability.

57. (*a*) Under this heading we include all resistances not covered by the previous investigation, namely the resistances encountered by the body of the machine and engine and the aeronaut, also tangential forces due to skin friction and so forth.

The most general supposition consistent with the

assumption that the pressure of the air on a moving body varies as the square of the relative velocity, consists in admitting that the effect of these resistances is represented by additional terms in X_o, Y_o, N_o proportional to U^2, and in the nine coefficients $X_u \ldots N_r$ proportional to U. We will for the present use accented letters $X'_o \ldots N'_r$ to represent these added terms, which, moreover, must satisfy the relations $X'_u = 2X'_o/U$, $Y'_u = 2Y'_o/U$, and $N'_u = 2N'_o/U$.

Now in the first place if Y'_o be different from zero, this represents an *additional* lift due to the air resistances, not accounted for by the action of the aeroplanes themselves; similarly, N'_o represents a couple due to this cause. It will, I think, be generally admitted that these quantities may be assumed to vanish in an ordinary machine, leaving only the resistance X'_o to be taken into account. At the same time, the following method shows that in *normal* circumstances they would have little effect on the stability.

In the first place, it is easy to show (*cf.* Lanchester, vol. I., chap. vii. § 164) that for an aeroplane of given weight the propeller thrust is a minimum when the head resistance is equal to the drift, while the horse-power is a minimum when the head resistance is one-third of the drift. In either case *the head resistance* X'_o *is comparable with the drift* $KS_1U^2 \sin^2 a$, *and is not large compared with it.* If it be necessary to consider flights performed at speeds largely in excess of the most economical speed, in which the head resistance is large compared with the drift, a fresh examination of the conditions of stability will be required. Except, however, at these excessive speeds, the head resistance X'_o will not be large compared with $KS_1U^2 \sin^2 a$, and the part X'_u will not be large compared with the portion $(2KS_1U \sin^2 a)$ of X_u due to the planes. In fact X'_u will be a small quantity of order a^2 (at least).

Of the remaining coefficients of the group $(X', Y', N')_{u, v, r}$, those which do not vanish will not in general be large compared with X_u, and, therefore,[1] they will also be at least of order a^2.

Bearing this fact in mind, and neglecting quantities of higher order where these are added to quantities of lower order, we obtain the approximate results,

$$\mathfrak{A} = W^2C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89a)$$

$$\frac{\mathfrak{B}}{gU} = KW\{(S_1 + S_2)C + Wl^2S_2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89b)$$

$$\frac{\mathfrak{C}}{g^2U^2} = KWS_2l\left\{KS_1l + \frac{W}{g}\right\} \text{ as in } (51c) \quad . \quad . \quad . \quad . \quad . \quad . \quad (89c)$$

$$\frac{\mathfrak{D}}{g^3U} = \frac{X'_u}{U}\left(K^2S_1\cos^2 a\, S_2l^2 + \frac{W}{g}KS_2l\right) + \frac{2W}{g}K^2S_1S_2l\sin^2 a + \frac{W^2}{gU^2}KS_2l\sin\theta \quad . \quad . \quad . \quad . \quad (89d)$$

$$\frac{\mathfrak{E}}{g^4U^4} = \frac{2W}{U^2g}K^2S_2l\left\{S_1\sin a\cos a\cos\theta + \left(S_1\sin^2 a + \frac{X'_u}{KU}\right)\sin\theta\right\} \quad (89e)$$

the terms involving the other coefficients $X'_u \ldots Y'_u \,.\, N'_r$ being of higher order than those retained and therefore negligible. It is thus seen that the coefficient X'_u alone enters into the lowest approximations, and thus has an important effect on the stability.

Now let the ratio of the head resistance X'_o to the drift $KS_1U^2\sin^2 a$, be denoted by v, then we shall have $X'_u = v \,.\, 2KS_1U\sin^2 a$, and by making use of the condition of equilibrium (83) the expression for $\mathfrak{D}$ simplifies down to

$$\frac{\mathfrak{D}}{g^3U^3} = \frac{W}{g}K^2S_1S_2l\sin a\cos a\left\{2(1+v)\tan a + \tan\theta + \frac{2vlg}{U^2}\cos\theta\right\} \quad (90)$$

while $\mathfrak{E}$ is to the lowest approximation given by

$$\frac{\mathfrak{E}}{g^4U^4} = \frac{2W}{U^2g}K^2S_1S_2l\sin a$$

(as in horizontal flight).

It will thus be seen that *the main effect of head resistance is to increase the coefficient $\mathfrak{D}$, thus increasing the*

[1] Any structure which gives rise to resistances not satisfying this condition would practically be of the nature of a "plane" and should be included among these.

stability by increasing the modulus of decay of the long oscillation. This result, which is best obtained in the first instance by algebra as is done above, is entirely in accordance with what we should expect from general considerations regarding the nature of the long and short oscillations as discussed in §§ 50—53.

Head resistance thus enables the machine to rise at a steeper angle to the horizon than would be possible if such resistance did not exist. For example, the effects of head resistance and inclination of flight path will cancel one another if

$$2v \tan a + \tan\theta + \frac{2vlg}{U^2}\cos\theta = 0 \quad . \quad . \quad . \quad (91)$$

or putting $\cos\theta = 1$,

$$\tan\theta = -2v\left(\tan a + \frac{lg}{U^2}\right) = -2v\left(\tan a + \frac{l}{2\,.\,\text{velocity height}}\right) \quad (91a)$$

If the machine is ascending at an angle θ given by this formula, it will be as stable as it would when moving horizontally in the absence of head resistance, and generally the increased stability due to head resistance enables the tangent of the inclination of the flight path to be increased by the amount approximately given above.

(*b*) In all the previous work the propeller thrust has been assumed constant, but if either the horse-power or the rate of revolution of the propeller is maintained constant the thrust will decrease when the velocity increases. In either case the effect of these variations can be represented by the addition of a positive term to X_u, and will be precisely similar to that of head resistance. It will be expressible in the form of an increase in the value of v defined above, and will increase the elevation of the flight path for given stability.

The effect of such variations when the propeller thrust does not pass through the centre of gravity can be investigated if the necessity arises, but the formulæ will necessarily become somewhat complicated.

58. Mr. E. H. Harper has expressed the condition of symmetric stability in terms of the forces, taking account of inclination of the flight path and also of head resistance. This condition may, of course, be easily modified to take account of variable propeller thrust, though in the following formula the thrust P is assumed constant :

Let

$$\begin{aligned} R_o &= \text{head resistance} \\ R_1 &= KU^2S_1 \sin^2 a = \text{component of drift} \\ P &= \text{propeller thrust} \end{aligned}$$

then by the conditions of equilibrium we have

$$W \sin \theta = R_o + R_1 - P \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (92)$$

and Mr. Harper finds that the approximate condition of stability $\mathfrak{E}\mathfrak{D} - \mathfrak{E}\mathfrak{B} > 0$ reduces to

$$\begin{aligned} 2R_o + \frac{U^2 \tan a}{gl \cos \theta}\left\{5R_o + R_1 - P - \frac{2k^2}{l^2}\,\frac{S_1 \cos^2 a + S_2}{S_2}\,R_1\right\} \\ + \frac{U^4 \tan^2 a}{g^2 l^2 \cos^2 \theta}\left\{3R_o + 3R_1 - P\right\} > 0 \quad . \quad . \quad . \quad . \qquad (93) \end{aligned}$$

For the case of horizontal flight without head resistance $R_o = 0$, $R_1 = P$, and the condition reduces at once to (54) or ($54a$).

CHAPTER VI

LONGITUDINAL STABILITY OF DOUBLE-LIFTING SYSTEMS. EXTENSION OF RESULTS TO SYSTEMS OTHER THAN NARROW PLANES MOVING AT SMALL ANGLES

The principle of equivalent systems.

59. We have hitherto only considered cases where the weight of the machine is entirely supported by the front plane or planes, the rear plane acting only as a tail, and being placed in a neutral direction.

We shall now show how *any results proved for such a*

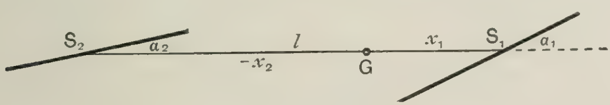

Fig. 26.

system can be extended to a system of two supporting planes placed tandem, and this we call a *double lifting system*. We assume the planes to be so near the axis of x that the principle of independence of height holds good ; at the same time that principle allows the front and rear planes to be "stepped," or placed at a somewhat different level, so that the results may be less modified by the "wash" which the front planes may produce on the back ones.

Let S_1, S_2 be the areas of the front and rear planes, a_1,

Photo.] [*The Sport and General Illustrations Co.*

FIG. II.—THE LATE M. E. LEFEBVRE FLYING AT RHEIMS.

The earlier type of Wright Biplane with auxiliary planes only in front of the main planes. Partial longitudinal stability dependent on the short oscillations is obtainable by placing the auxiliary planes sufficiently in advance of the main planes (§ 51). To secure longitudinal stability for the long oscillations as well, the front planes must be tilted up at a greater inclination than those at the rear; the aeroplane thus becomes a "double lifting system," as discussed in § 60.

a_2 their angles of attack, x_1, x_2 the co-ordinates of their centres of pressure. Using the " S', μ " notation,

$$S'_1 = S_1 \cos^2 a, \quad S'_2 = S_2 \cos^2 a_2. \quad \mu_1 = \tan a_1, \quad \mu_2 = \tan a_2,$$

we see that the resistance derivatives can be written in the form

$$\begin{array}{llll} X_o/KU^2 = \Sigma S'\mu^2 & X_u/KU = 2\Sigma S'\mu^2 & X_v/KU = \Sigma S'\mu & X_r/KU = \Sigma S'\mu x \\ Y_o/KU^2 = \Sigma S'\mu & Y_u/KU = 2\Sigma S'\mu & Y_v/KU = \Sigma S' & Y_r/KU = \Sigma S'x \\ N_o/KU^2 = \Sigma S'\mu x & N_u/KU = 2\Sigma S'\mu x & N_v/KU = \Sigma S'x & N_r/KU = \Sigma S'x^2 \end{array} \quad (94)$$

These involve the six sums,

$$\Sigma S', \ \Sigma S'\mu, \ \Sigma S'\mu^2, \ \Sigma S'x, \ \Sigma S'\mu x, \ \Sigma S'x^2$$

the values of which depend on the values of the six quantities, S'_1, S'_2, μ_1, μ_2, x_1, x_2.

The six sums are, however, not independent in virtue of the determinantal relation

$$\frac{\Delta_o}{2K^3U^3} = \begin{vmatrix} \Sigma S'\mu^2 & \Sigma S'\mu & \Sigma S'\mu x \\ \Sigma S'\mu & \Sigma S' & \Sigma S'x \\ \Sigma S'\mu x & \Sigma S'x & \Sigma S'x^2 \end{vmatrix} = 0,$$

which has been proved for systems with two planes in § 26.

If follows, therefore, that if any two systems have *five* of the six sums equal, the remaining sums will be equal, and the conditions of equilibrium and stability of the two systems will be identical. Such systems will be described as *equivalent*.

A tandem system of two planes can thus be replaced by an equivalent system satisfying one additional arbitrary condition. If we assume as this condition $\mu_2 = 0$, the equivalent system will have its rear plane in the neutral position, and the connections between the two may be given in the following table :

	Double Lifting System	Coefficients	Equivalent System
(i)	$S_1 \sin^2 \alpha_1 + S_2 \sin^2 \alpha_2$	$\frac{X_o}{KU^2}$ and $\frac{X_u}{2KU}$	$S_1 \sin^2 \alpha$
(ii)	$S_1 \sin \alpha_1 \cos \alpha_1 + S_2 \sin \alpha_2 \cos \alpha_2$	$\frac{Y_o}{KU^2}, \frac{Y_u}{2KU}, \frac{X_v}{KU}$	$S_1 \sin \alpha \cos \alpha$
(iii)	$S_1 x_1 \sin \alpha_1 \cos \alpha_1 + S_2 x_2 \sin \alpha_2 \cos \alpha_2$	$\frac{N_o}{KU^{r}}, \frac{N_u}{2KU}, \frac{X_r}{KU}$	$S_1 x_1 \sin \alpha \cos \alpha$
(iv)	$S_1 \cos^2 \alpha_1 + S_2 \cos^2 \alpha_2$	$\frac{Y_v}{KU}$	$S_1 \cos^2 \alpha + S_2$
(v)	$S_1 x_1 \cos^2 \alpha_1 + S_2 x_2 \cos^2 \alpha_2$	$\frac{N_v}{KU}, \frac{Y_r}{KU}$	$S_1 x_1 \cos^2 \alpha + S_2 x_2$
(vi)	$S_1 x_1^2 \cos^2 \alpha_1 + S_2 x_2^2 \cos^2 \alpha_2$	$\frac{N_r}{KU}$	$S_1 x_1^2 \cos^2 \alpha + S_2 x_2^2$

(95)

We have supposed for the sake of greater generality that the propeller thrust does *not* necessarily pass through the centre of gravity. If it does, $N_o = 0$, $x_1 = 0$, $x_2 = -l$ where l is the distance of the tail plane behind the centre of pressure of the main plane, and each member of (iii) vanishes.

It will be noticed, by adding (i) and (iv), that the total superficial area $S_1 + S_2$ of the surfaces is the same in the two equivalent systems. In the S', μ notation the area is $\Sigma S'(1 + \mu^2)$.

The **principle of equivalence** thus established can now be stated as follows :

To every double lifting system of two narrow planes there corresponds an equivalent single lifting system having the same conditions of equilibrium and stability, and the same total superficial area of its planes.

The conditions of stability of a double lifting system are usually considerably simplified by replacing it by the equivalent single lifting system. Without using this artifice the algebra is very long and cumbersome.

Example.—Let $\alpha_1 = 20^\circ$, $\alpha_2 = 10^\circ$, $S_1 = 20$, $S_2 = 10$ units of area, and let $x_1 - x_2$, the distance between the centres of pressure, be equal to 10 units of length. Let the propeller

thrust be central; then the equation of moments $S_1 a_1 x_1 + S_2 a_2 x_2 = 0$ gives $4x_1 + x_2 = 0$, whence $x_1 = 2$, $x_2 = -8$ in the original system.

For the equivalent system we have $x_1 = 0$, $x_2 = -l$, where l is the distance of the tail plane, and the remaining equations give, remembering that the angles are small,

$$\begin{array}{rcllcl}
S_1 a^2 & = & 20.20^2 & + \ 10.10^2 & = & 9000 \quad (a) \\
S_1 a & = & 20.20 & + \ 10.10 & = & 500 \quad (b) \\
S_1 + S_2 & = & 20 & + \ 10 & = & 30 \quad (c) \\
-S_2 l & = & 20.2 & - \ 10.8 & = & -40 \quad (d) \\
S_2 l^2 & = & 20.2^2 & + \ 10.8^2 & = & 720 \quad (e)
\end{array}$$

From (a) and (b) we find $a = 18^\circ$, $S_1 = 27\frac{7}{9}$, hence by (c) $S_2 = 2\frac{2}{9} = \frac{20}{9}$, and by substitution in (d) $l = 18$, which agrees with the result obtained by dividing (e) by (d).

Thus, in the equivalent system, the areas of the supporting and tail planes are $27\frac{7}{9}$ and $2\frac{2}{9}$ units; the inclination of the former is 18°, and the distance between the planes is 18 units of length.

Invariants of equivalent systems.

60. In deducing the conditions of stability of a tandem or double lifting system from those of its equivalent single system a great deal of time may still be wasted in algebraical complications, and this was done in Mr. Harper's and my early solutions. A considerable simplification may be made by first making a list of the *invariants* of equivalent systems as follows:

(1) If

$$\left.\begin{array}{l} d = -\Sigma S'x/\Sigma S' \\ f = -\Sigma S'\mu x/\Sigma S'\mu \\ R^2 = \Sigma S'x^2/\Sigma S' \end{array}\right\} \quad . \quad . \quad . \quad . \quad (96)$$

then d, f and R are invariants. Of these d represents the distance behind the origin of the centre of gravity of S'_1 and S'_2, f the distance of the centre of pressure of the planes themselves, R the radius of gyration of S'_1, S'_2

about the origin. The inclinations α_1, α_2 being small, d and R may to a sufficient degree of approximation be assumed to refer to the areas S_1, S_2 instead of S'_1 and S'_2.

(2) The minors of the determinant in Δ_o are also invariants, and these are

$$\left.\begin{array}{lll} S'_1S'_2(\mu_1 - \mu_2)^2 & \text{Minor of} & \Sigma S'x^2 \\ -S'_1S'_2(\mu_1 - \mu_2)(x_1 - x_2) & \text{,,} \quad \text{,,} & \Sigma S'\mu x \\ S'_1S'_2(x_1 - x_2)^2 & \text{,,} \quad \text{,,} & \Sigma S'\mu^2 \\ -S'_1S'_2(\mu_1 - \mu_2)(\mu_1x_2 - \mu_2x_1) & \text{,,} \quad \text{,,} & \Sigma S'x \\ S'_1S'_2(x_1 - x_2)(\mu_1x_2 - \mu_2x_1) & \text{,,} \quad \text{,,} & \Sigma S'\mu \\ S'_1S'_2(\mu_1x_2 - \mu_2x_1)^2 & \text{,,} \quad \text{,,} & \Sigma S' \end{array}\right\} \quad . \qquad (97)$$

so that the ratios $\mu_1 - \mu_2 : x_1 - x_2 : \mu_1x_2 - \mu_2x_1$ are also invariants. It will be noticed that $x_1 - x_2$ is the distance between the front and rear planes, which may be denoted in every case by l, and that $S'_1S'_2(\mu_1 - \mu_2)^2 = S_1S_2 \sin^2(\alpha_1 - \alpha_2)$ where $\alpha_1 - \alpha_2$ is the angle between the directions of the front and rear planes. This angle we shall denote by α, and we shall further write $\mu_1 - \mu_2 = \nu$ so that $\nu = \alpha$ approximately, the angles being small.

61. To extend the conditions of stability of a single lifting system to a double lifting one, the other conditions of the problem being the same, it is only necessary to express the inequalities representing them in terms of invariants.

Taking the "simplest case" of § 48, where the direction of flight is horizontal, the propeller thrust is also in the same direction and passes through the centre of gravity, we may write (58),

$$\frac{U^2}{g} > k^2 \frac{l}{\mu} \frac{S'_1 + S_2}{S_2 l^2}$$

l/μ is an invariant and $= l/\nu$ or $(x_1 - x_2)/(\mu_1 - \mu_2)$ for the compound system.

$S_2l^2/(S'_1 + S_2)$ is also the invariant $\Sigma S'x^2/\Sigma S'$ or R^2 as defined above, and the condition becomes

$$\frac{U^2}{g} > k^2 \frac{l}{\nu R^2} \text{ or } k^2 \frac{x_1 - x_2}{\mu_1 - \mu_2} \frac{S'_1 + S'_2}{S'_1x^2_1 + S'_2x^2_2} \quad . \quad . \qquad (98)$$

Taking the other form, we may write it

$$\frac{W}{Kg} > k^2\, S'_1 S_2 l^2 \frac{S'_1 + S_2}{S_2 l S_2 l^2}$$

The invariants here are

$$\begin{aligned} S'_1 S_2 l^2 &= S'_1 S'_2 (x_1 - x_2)^2 \text{ or again } S'_1 S'_2 l' \\ S'_1 + S_2 &= S'_1 + S'_2 \\ S_2 l &= -\Sigma S' x = (S'_1 + S'_2) d \\ S_2 l^2 &= \Sigma S' x^2 = (S'_1 + S'_2) R^2 \end{aligned}$$

and we may write the condition

$$\frac{W}{Kg} > k^2 \frac{S'_1 S'_2}{S'_1 + S'_2} \frac{l^2}{dR^2} \quad . \quad . \quad . \quad . \quad . \quad (99)$$

or in a number of different other ways; for example, we may write

$$\frac{W}{Kg} > k^2 \frac{(S'_1 \mu)^2 (S'_1 + S_2)}{S'_1 \mu^2\, S_2 l}$$

which reduces by invariants to

$$\frac{W}{Kg} > k^2 \frac{(\Sigma S' \mu)^2\, \Sigma S'}{\Sigma S' \mu^2 (-\Sigma S' x)} \text{ or } \frac{k^2}{d} \frac{(\Sigma S' \mu)^2}{\Sigma S' \mu^2} \quad . \quad . \quad . \quad (100)$$

It is further to be pointed out that the condition $\nu_l l$ positive, which is itself in invariant form, shows that *the front plane must have the larger angle of attack.* It is easy to waste time in turning and twisting the conditions of stability into various other forms, and much of this danger is saved by a clear understanding of the invariants.

The propeller thrust does not pass through the centre of gravity.

62. If the propeller thrust does not pass through the centre of gravity it is obvious by moments or otherwise that if the tail plane is in a neutral direction the centre of pressure of the front plane cannot coincide with the centre of gravity. Taking, however, a double lifting system, supposed descending at an angle θ with the horizon and assuming H the propeller thrust, h its distance

from the origin (Fig. 27), η its inclination to the line of flight, the conditions of equilibrium become, with the usual notation

$$\begin{aligned} KU^2\Sigma S'\mu^2 &= W \sin\theta + H \cos\eta \\ KU^2\Sigma S'\mu &= W \cos\theta + \underline{H \sin\eta} \\ KU^2\Sigma S'\mu x &= -Hh \quad . \quad . \quad . \quad . \quad . \quad . \qquad (101) \end{aligned}$$

In practice the angles θ and η will be small, and from the first equation it is evident that H is then small compared with W, and still more is $H \sin\eta$ compared with $W \cos\theta$. The term underlined will therefore be neglected in what follows, and we will further assume that $\cos\eta$ may

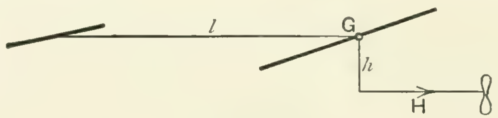

Fig. 27.

be put equal to unity. This is equivalent to assuming either that $\eta = 0$, or that even if η is not zero, the corrections discussed in § 54 are negligible.

The third equation shows, as stated above, that the system cannot be replaced by an equivalent system with both $x_1 = 0$ and $\mu_2 = 0$. All we can do to simplify the algebra is to assume an equivalent system where either $x_1 = 0$ *or* $\mu_2 = 0$. Which is the better choice? Algebraically the simplest plan would be to assume the rear plane to be neutral or $\mu_2 = 0$, the third condition then giving $KS'_1U^2x_1\mu_1 = -Hh$. For the purpose of applications the reverse assumption $x_1 = 0$ seems on the whole most desirable, giving $KS'_2U^2\mu_2x_2 = -Hh$. For if H is variable, or, indeed, H becomes zero through stopping the engines, the conditions of equilibrium are modified, and it is clear that the easiest way of adjusting equilibrium is by turning the tail plane so as to alter μ_2. On the other hand, if the adjustment were made by varying x_1, this would involve shifting the centre of gravity of the machine, thereby

probably affecting the moments of inertia, and certainly effecting changes in the equivalent system that would be less easy to take into proper account.

Assuming, therefore, for the equivalent system $x_1 = 0$ and $x_2 = -l$, the determinantal equation in λ becomes

$$\begin{vmatrix} 2(S'_1\mu_1^2 + S'_2\mu_2^2) + \dfrac{W\lambda}{KUg}, & S'_1\mu_1 + S'_2\mu_2, & -S'_2\mu_2 l - \dfrac{W\cos\theta}{KU\lambda} \\ 2(S'_1\mu_1 + S'_2\mu_2), & S'_1 + S'_2 + \dfrac{W\lambda}{KUg}, & -S'_2 l + \dfrac{W}{Kg} + \dfrac{W\sin\theta}{KU\lambda} \\ -2S'_2\mu_2 l, & -S'_2 l, & +S'_2 l^2 + \dfrac{Wk^2\lambda}{KUg} \end{vmatrix} = 0 \quad . \quad . \quad (102)$$

The coefficients of the biquadratic are given by

$$\frac{\mathfrak{A}}{W^3} = k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (103a)$$

$$\frac{\mathfrak{B}}{W^3}\left(\frac{W}{KUg}\right) = k^2(S'_1 + S'_2) + S'_2 l^2 + \underline{2k^2\Sigma S'\mu^2} \quad . \quad . \quad . \quad (103b)$$

$$\frac{\mathfrak{C}}{W^3}\left(\frac{W}{KUg}\right)^2 = S'_1 S'_2 l^2 + \frac{W}{Kg}S'_2 l + \underline{2k^2 S'_1 S'_2(\mu_1 - \mu_2)^2} + \underline{2S'_1 S'_2\mu_1^2 l^2} \quad (103c)$$

$$\frac{\mathfrak{D}}{W^3}\left(\frac{W}{KUg}\right)^3 = \frac{2W}{Kg}S'_1 S'_2(\mu_1 - \mu_2)\mu_1 l$$
$$+ \frac{W^2}{K^2U^2g}S'_2 l(-2\mu_2\cos\theta + \sin\theta) \quad . \quad . \quad . \quad . \quad (103d_1)$$

$$\frac{\mathfrak{E}}{W^3}\left(\frac{W}{KUg}\right)^4 = \frac{2W^2}{K^2U^2g}S'_1 S'_2 l(\mu_1 - \mu_2)(\cos\theta + \underline{\mu_1\sin\theta}) \quad . \quad . \quad (103e_1)$$

where the terms underlined are small (being of order μ^2 if θ is comparable with μ) in comparison with those retained, and they may therefore be neglected.

We have, further, by the approximate conditions of equilibrium (101)

$$W\cos\theta = KU^2(S'_1\mu_1 + S'_2\mu_2)$$

whence

$$\frac{\mathfrak{E}}{W^3}\left(\frac{W}{KUg}\right)^4 = \frac{2W}{Kg}S'_1 S'_2 l(\mu_1 - \mu_2)(S'_1\mu_1 + S'_2\mu_2) \quad . \quad . \quad . \quad . \quad (103e_2)$$

$$\frac{\mathfrak{D}}{W^3}\left(\frac{W}{KUg}\right)^3 = \frac{2W}{Kg}S'_2 l\left\{S'_1\mu_1^2 - S'_1\mu_1\mu_2 - (S'_1\mu_1 + S'_2\mu_2)(\mu_2 - \tfrac{1}{2}\tan\theta)\right\}$$
$$= \frac{2W}{Kg}S'_2 l\left\{(S'_1\mu_1 + S'_2\mu_2)(\mu_1 - \mu_2 + \tfrac{1}{2}\tan\theta) - (S'_1 + S'_2)\mu_1\mu_2\right\}$$
$$(103d_2)$$

When $\mathfrak{D}$ is expressed in this form, it will be seen that $S'_1\mu_1 + S'_2\mu_2$ occurs as a factor, not only of $\mathfrak{E}$, but also of the whole of $\mathfrak{D}$ except the last term, which vanishes when $\mu_2 = 0$, as occurs in the case of $h = 0$. Writing $\mu_1 - \mu_2 = \nu$, we now find that the condition $\mathfrak{E}\mathfrak{D} - \mathfrak{E}\mathfrak{B} > 0$, when reduced to its simplest form, becomes

$$\frac{W}{Kg}S'_2 l - k^2(S'_1 + S'_2)S'_1 + \left(S'_1 S'_2 l^2 + \frac{W}{Kg} S'_2 l \right)\left\{\frac{\tan\theta}{2\nu} - \frac{(S'_1 + S'_2)\mu_1\mu_2}{\nu(S'_1\mu_1 + S'_2\mu_2)}\right\} > 0 \quad . \quad (104)$$

When we transform this inequality by means of invariants to cover the case of a system where x_1 is different from zero, we may use the conditions of equilibrium,

$$S'_2\mu_2 l = \frac{Hh}{KU^2} = \frac{Hh}{W\cos\theta}(S'_1\mu_1 + S'_2\mu_2)$$

and the invariants of § 60, including the notation,

$$\begin{aligned} S &= S'_1 + S'_2, \\ Sd &= -(S'_1 x_1 + S'_2 x_2) = S'_2 l, \\ SR^2 &= S'_1 x_1^2 + S'_2 x_2^2 = S'_2 l^2 \end{aligned}$$

and we may express the result in a variety of forms such as

$$\frac{W}{Kg}Sd - k^2 S'_1 S'_2 \frac{l^2}{R^2} + \left(S'_1 S'_2 l^2 + \frac{W}{Kg} Sd \right)\frac{ld}{\nu R^2}\left\{\frac{\tan\theta}{2} - \frac{Hh}{W\cos\theta}\,\frac{\mu_2 x_1 - \mu_1 x_2}{ld}\right\} > 0 \quad . \quad (105)$$

63. Mr. Harper has verified this condition by an independent method, starting with a system in which the tail is neutral, but the front plane has its centre of pressure not coinciding with the centre of gravity. In this case, $\mu_2 = 0$, but x_1 is different from 0; his condition is

$$\begin{aligned} &\frac{W}{Kg}(S'^2_2 x_2^2 - S'^2_1 x_1^2) - S'_1 S'_2 (x_1 - x_2)(S'_1 + S'_2)(k^2 + x_1 x_2) \\ &+ \frac{\tan\theta}{2\mu_1}(S'_1 x_1 + S'_2 x_2)\left[\frac{W}{Kg}(S'_1 x_1 + S'_2 x_2) - S'_1 S'_2 (x_1 - x_2)^2\right] > 0 \quad (106) \end{aligned}$$

which reduces to the above as he has verified.

He has also obtained the following form for this case:

$$\frac{W}{Kg}\left(\frac{d^2}{S'_1} - \frac{2fd}{S'_1 + S'_2}\right) - k^2(d-f) - f(R^2 - df)$$
$$+ \frac{\tan\theta}{2\mu} d\left[\frac{W}{Kg}\frac{d}{S'_1} + R^2 + f^2 - 2fd\right] > 0 \quad . \quad . \quad . \quad . \quad . \quad (107)$$

where, as in § 60, f is the distance of the centre of pressure behind the centre of gravity, S'_1 is equal to the invariant $(\Sigma S'_1\mu)^2/'\Sigma S'\mu^2$, and

$$\frac{S'_1 + S'_2}{S'_1} = \frac{R^2 + f^2 - 2fd}{R^2 - d^2} \quad . \quad . \quad . \quad (108)$$

It has been necessary also to work through a number of other formulæ which are not of sufficient interest to justify their publication at the present time.

As H is small compared with W, it easily follows that unless h is large compared with l the correction for non-central propeller thrust will be small. In an actual flying machine h would certainly be small in comparison with l.

The above formulæ seem to suggest that making h positive, or lowering the propeller thrust, tends to decrease the stability, but since any change of this kind also affects the conditions of equilibrium, the question is not a definite one until it is clearly specified what other changes in the system are effected in order to balance the moment of the thrust. If we suppose the balance to be restored by moving the centre of gravity forward, or, what is the same thing, shifting the planes both backwards through a small distance, we may show from (106) *that in the case of horizontal flight, stability is increased by lowering the propeller axis and decreased by raising it if*

$$2k^2 > \left(\frac{\Sigma S'x^2}{\Sigma S'x}\right)^2 \text{ or } \left(\frac{R^2}{d}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

but the corresponding converse when (109) is not satisfied is less simple. In this case, lowering the propeller axis

decreases the stability if this stability is small and increases it if already considerable.

A case of more frequent occurrence in flying machines is that in which the propeller axis and the planes are about the same level and the centre of gravity is below them. This is equivalent to raising the propeller axis and the planes simultaneously above the centre of gravity, so that the moments of the drift and propeller thrust continue to balance each other. It will be seen that in this case the "principle of independence of height" must no longer be assumed, the corrections to be applied to the formulæ depending essentially on the small deviations from that property.

Broad Planes.
Effect of shifting of centre of pressure.

64. We shall now suppose that when the angle of attack α is increased by $d\alpha$, the co-ordinate of the centre of pressure of a surface S increases by $a\phi'(\alpha)d\alpha$ as in § 23; this will in general be negative; thus taking Joessel and Avanzini's formula, the distance of the centre of pressure from the centre of figure $= 0{\cdot}6a(1 - \sin\alpha) = 0{\cdot}3(1 - \sin\alpha) \times$ (whole breadth $2a$), and thus

$$a\phi'(\alpha) = -0{\cdot}6\, a \cos\alpha \quad . \quad . \quad . \quad . \quad (110)$$

It will be convenient to write

$$a\phi'(\alpha) = -e\cos\alpha \quad . \quad . \quad . \quad . \quad . \quad (111)$$

and e will then represent, *according to this formula*, the distance of the centre of pressure in front of the centre of area at "grazing incidence" when the angle of attack just vanishes. If any other formula for the centre of pressure be assumed, we need only *assume* e *to be a function of* α *defined by* $a\phi'(\alpha) = -e\cos\alpha$. We

do not in this section take account of the "rotary coefficients" of § 23.

The effect of the shifting is to alter the coefficient N_v to $KU\Sigma S'(x-e\mu)$ and N_r to $KU\Sigma S'(x-e\mu)x$.

In this section we shall assume that the propeller thrust acts along the direction of steady flight and passes through the centre of gravity, *i.e.* we neglect the corrections of §§ 54, 62.

Now in a single lifting system with neutral tail x vanishes for the front plane and μ for the rear plane, so that $\Sigma S'e\mu x = 0$, that is, the part added to N_r vanishes. This is not necessarily the case for a doubly lifting or tandem system. For this reason such a system does not in general possess a corresponding singly lifting system which is equivalent to it in every respect both algebraically and physically. An exception occurs when e is the same for the front and rear planes, so that $\Sigma S'e\mu x = e\Sigma S'\mu x = 0$ by the equation of moments. We may, however, still simplify the algebra very greatly by replacing the doubly lifting system by a singly lifting one which is equivalent to it in every respect but the presence of the added term in N_r, that is, we employ the substitutions of § 59, with the additional substitution of $S'_1e\mu$ for $\Sigma S'e\mu$ (this determines the value of e), *but with the retention of the additional term* $-\Sigma S'e\mu x$ in the third column, bottom row, of the determinant.

[I have worked out the solution without the use of an equivalent system, but the algebra is very much longer, and it is therefore desirable to use this short cut.]

The determinantal equation in λ is now

$$\begin{vmatrix} 2S'_1\mu^2 + \dfrac{W\lambda}{KUg}, & S'_1\mu, & -\dfrac{W\cos\theta}{KU\lambda} \\ 2S'_1\mu, & S'_1 + S_2 + \dfrac{W\lambda}{KUg}, & -S_2l + \dfrac{W}{Kg} + \dfrac{W\sin\theta}{KU\lambda} \\ 0, & -S_2l - S'_1e\mu, & S_2l^2 - \Sigma S'e\mu x + \dfrac{Wk^2\lambda}{KUg} \end{vmatrix} = 0 \qquad (112)$$

Neglecting terms of order μ^2 in the first three coefficients

and making use of the equilibrium condition $W \cos\theta = KU^2 S'_1\mu$ we get

$$\frac{\mathfrak{A}}{W^3} = k^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113a)$$

$$\frac{\mathfrak{B}}{W^3}\left(\frac{W}{KUg}\right) = k^2(S'_1 + S_2) + \underline{S_2 l^2 - \Sigma S' e\mu x} \quad . \quad . \quad . \quad (113b)$$

$$\frac{\mathfrak{C}}{W^3}\left(\frac{W}{KUg}\right)^2 = \underline{S'_1 S_2 l^2 - S'_1 \Sigma S' e\mu x} - S_2(S'_1 e\mu l + \Sigma S' e\mu x) \quad . \quad . \quad (113c)$$
$$+ \frac{W}{Kg}(S_2 l + S'_1 e\mu)$$

$$\frac{\mathfrak{D}}{W^3}\left(\frac{W}{KUg}\right)^3 = -\underline{\underline{2S'_1 S_2 \mu^2(S'_1 e\mu l + \Sigma S' e\mu x)}} + \frac{W}{Kg}(S_2 l + S'_1 e\mu)S'_1\mu(\underline{2\mu} + \tan\theta)$$
$$(113d)$$

$$\frac{\mathfrak{E}}{W^3}\left(\frac{W}{KUg}\right)^4 = \frac{2W}{Kg}(S_2 l + S'_1 e\mu)S'^2_1\mu^2(\underline{1} + \mu\tan\theta) \quad . \quad . \quad . \quad (113e)$$

In the approximate condition $\mathfrak{C}\mathfrak{D} - \mathfrak{E}\mathfrak{B} > 0$ the products of the terms once underlined cancel; and in $\mathfrak{E}$ we may neglect $\mu \tan\theta$ compared with unity. *For horizontal flight* the condition reduces to

$$- k^2 S'_1(S'_1 + S_2) + \frac{W}{Kg}(S_2 l + S'_1 e\mu) - 2S_2(S'_1 e\mu l + \Sigma S' e\mu x)$$
$$- \frac{Kg}{W}\,\frac{\{S'_1 S_2(l^2 - e\mu l) - (S'_1 + S_2)\Sigma S' e\mu x\}\{S'_1 e\mu l + \Sigma S' e\mu x\}S_2}{S_2 l + S'_1 e\mu}$$
$$> 0 \quad . \quad . \quad . \quad (114)$$

65. For a double lifting system this expression is easily transformed into the correct formula by means of invariants. In this case

$$S_2 l + S'_1 e\mu \text{ becomes } \Sigma S'(e\mu - x)$$

while to transform $S'_1 e\mu l + \Sigma S' e\mu x$ we observe that the term twice underlined in $\mathfrak{D}$ arises from the determinant

$$\begin{vmatrix} 2S'_1\mu^2, & S'_1\mu, & 0 \\ 2S'_1\mu, & S'_1 + S_2, & -S_2 l \\ 0, & -S'_1 e\mu, & -\Sigma S' e\mu x \end{vmatrix} \text{ which becomes } \begin{vmatrix} 2\Sigma S'\mu^2, & \Sigma S'\mu, & \Sigma S'\mu x \\ 2\Sigma S'\mu, & \Sigma S', & \Sigma S' x \\ 0, & -\Sigma S' e\mu, & -\Sigma S' e\mu x \end{vmatrix}$$

Now if the bottom term in the first column had been $-2\Sigma S' e\mu^2$ instead of zero the determinant would have vanished, and hence we find that

$$
\begin{aligned}
& - 2S'_1S'_2\mu^2(S'_1e\mu l + \Sigma S'e\mu x) \\
&= \begin{vmatrix} 2\Sigma S'\mu^2 & \Sigma S'\mu & \Sigma S'\mu x \\ 2\Sigma S'\mu & \Sigma S' & \Sigma S'x \\ 2\Sigma S e\mu^2 & 0 & 0 \end{vmatrix} \\
&= 2\Sigma S'e\mu^2\Sigma S'\mu\Sigma S'x \text{ (with } \Sigma S'\mu x = 0\text{) or generally} \\
&= 2\Sigma S'e\mu^2 S'_1S'_2(\mu_1 - \mu_2)(x_2 - x_1) \\
&= - 2\Sigma S'e\mu^2 S'_1S'_2\nu l \text{ with } \nu = \mu_1 - \mu_2 \text{ and } l = x_1 - x_2 \quad . \quad (115)
\end{aligned}
$$

It is not worth while to reproduce all the algebra that has been gone through in our preliminary investigations, as in all cases likely to be of interest considerable simplifications occur. For any system with two planes $e\mu$ will be small compared with l, and only first powers of e need be retained. It is only in the case of a single plane without a tail of any kind that this might become inadmissible, and then the algebra becomes very simple indeed. We consider only the following cases:

66. CASE I.—**Single lifting system with neutral tail.**—In this case $\Sigma S'e\mu x = 0$, and in the case of horizontal flight the condition reduces to

$$\frac{W}{Kg}(S_2l + S'_1e\mu) - k^2S'_1(S'_1 + S_2) - 2S'_1S_2e\mu l - \frac{Kg}{W}S'^2_1S_2e\mu l^2 > 0 \quad . \quad (116)$$

and the condition when higher powers of $e\mu$ are retained is not much more complicated.

For flight at an angle θ downwards with the horizon, if $\mu \tan\theta$ be neglected in $\mathfrak{E}$ we get

$$
\begin{aligned}
&\frac{W}{Kg}(S_2l + S'_1e\mu)\left(1 + \frac{\tan\theta}{2\tan a}\right) - k^2S'_1(S'_1 + S_2) + S'_1S_2l^2\frac{\tan\theta}{2\tan a} \\
&\qquad - S'_1S_2e\mu l\left\{2 + \frac{\tan\theta}{2\tan a} + \frac{Kg}{W}S'_1l\right\} > 0 \quad . \quad . \quad (117)
\end{aligned}
$$

Putting $W = KU^2S'_1\mu\sec\theta$, we get

$$
\begin{aligned}
&\frac{U^2}{g\cos\theta}\mu(S_2l + S'_1e\mu)\left(1 + \frac{\tan\theta}{2\tan a}\right) - k^2(S'_1 + S_2) + S_2l^2\frac{\tan\theta}{2\tan a} \\
&\qquad - S_2e\mu l\left\{2 + \frac{\tan\theta}{2\tan a} + \frac{g\cos\theta\, l}{U^2}\right\} > 0 \quad . \quad . \quad (118)
\end{aligned}
$$

a quadratic in U^2/g.

67. CASE II.—**Single plane without tail or rudder.**—This case may be deduced from the preceding one by putting S_2 and $l=0$. But where the stability is entirely dependent on the shifting of the centre of pressure, it is really easier to start afresh and use the complete

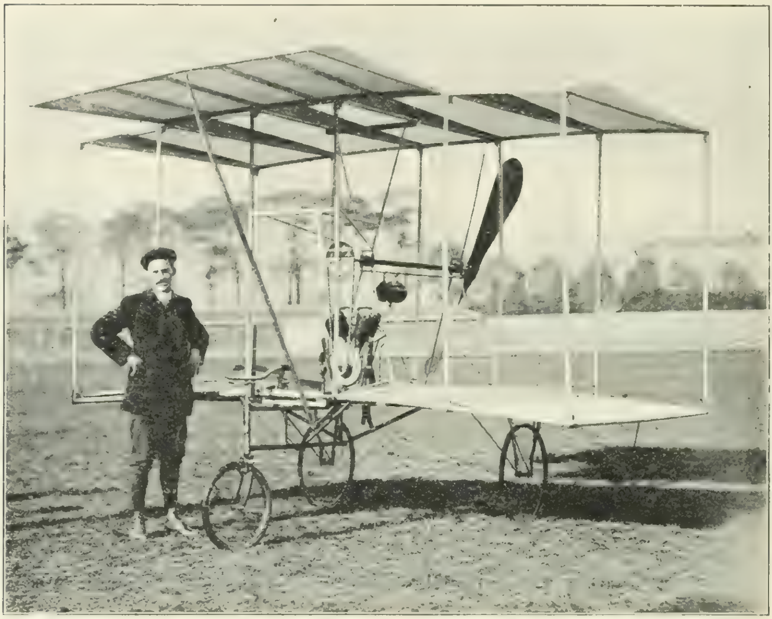

Photo.] [*The Sport and General Illustrations Co.*

FIG. III.—EDWARD MINES BIPLANE AT DONCASTER.

An aeroplane having no auxiliary tail planes or vertical fins. *Longitudinal stability* could only be attained by means of the shift of the centre of pressure with varying angle of attack (§ 67), and the stability condition would thus impose a superior limit to the speed. There is no *lateral stability*, all the coefficients in the biquadratic except the first two vanishing.

discriminant $\mathfrak{H}>0$ instead of the approximate substitute $\mathfrak{C}\mathfrak{D}-\mathfrak{E}\mathfrak{B}>0$. We have

$$\frac{\mathfrak{A}}{W^3}=k^2,\quad \frac{\mathfrak{B}}{W^3}\left(\frac{W}{KUg}\right)=S'k^2(1+2\mu^2)\quad \frac{\mathfrak{C}}{W^3}\left(\frac{W}{KUg}\right)^2=\frac{W}{Kg}S'e\mu$$

$$\frac{\mathfrak{D}}{W^3}\left(\frac{W}{KUg}\right)^3=\frac{2W}{Kg}S'^2e\mu^3\left(1+\frac{\tan\theta}{2\mu}\right)$$

$$\frac{\mathfrak{E}}{W^3}\left(\frac{W}{KUg}\right)^4=\frac{2W^2\cos\theta}{K^2U^2g}S'^2e\mu^2(1+\mu\tan\theta)\qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

whence

$$\frac{\mathfrak{B}\mathfrak{C} - \mathfrak{A}\mathfrak{D}}{W^6}\left(\frac{W}{KUg}\right)^3 = \frac{W}{Kg}S'^2 e\mu k^2(1 - \mu \tan \theta)$$

and with this substitution the condition which may be written

$$\mathfrak{H} \equiv \mathfrak{D}(\mathfrak{B}\mathfrak{C} - \mathfrak{A}\mathfrak{D}) - \mathfrak{E}\mathfrak{B}^2 > 0$$

reduces to

$$\frac{W}{Kg}e\mu(1 - \mu \tan \theta)\left(1 + \frac{\tan \theta}{2\mu}\right) - k^2 S'(1 + 2\mu^2)^2(1 + \mu \tan \theta) > 0 \quad . \quad (120)$$

that is

$$\frac{U^2}{g} > \frac{k^2(1 + 2\mu^2)^2 \cos \theta(1 + \mu \tan \theta)}{e\mu^2(1 - \mu \tan \theta)\left(1 + \dfrac{\tan \theta}{2\mu}\right)} \quad . \quad . \quad . \quad (121)$$

or going back to the substitutions $S' = S \cos {}^2 a$, $\mu = \tan a$.

$$\frac{U^2}{g} > \frac{k^2 \cos \theta(1 + 2 \tan {}^2 a)^2(1 + \tan a \tan \theta)}{e \tan a(1 - \tan a \tan \theta)(\tan a + \frac{1}{2} \tan \theta)} \quad . \quad (121a)$$

For *horizontal flight* $(\theta = 0)$ this becomes

$$\frac{U^2}{g} > \frac{k^2(1 + 2 \tan {}^2 a)^2}{e \tan {}^2 a}, \text{ or approximately } \frac{U^2}{g} > \frac{k^2}{e \tan {}^2 a} \quad . \quad (122)$$

or again

$$\frac{W}{Kg} > \frac{k^2 S(1 + \sin {}^2 a)^2}{e \sin a \cos a} \quad . \quad . \quad . \quad . \quad (122a)$$

For a glider $\theta = a$ giving

$$\frac{U^2}{g} > \frac{2}{3} \frac{k^2 \cos a\,(1 + 2 \tan {}^2 a)^2(1 + \tan {}^2 a)}{e \tan {}^2 a(1 - \tan {}^2 a)} \text{ or } \frac{2}{3} \frac{k^2(1 + 2 \tan {}^2 a)^2 \cos a}{e \sin {}^2 a(1 - \tan {}^2 a)} \quad (123)$$

or approximately

$$\frac{U^2}{g} > \frac{2}{3} \frac{k^2}{e \sin {}^2 a} \quad . \quad . \quad . \quad . \quad . \quad (124)$$

thus the minimum value of U^2/g required for the stability of a glider is only about two-thirds of that required for the same plane when driven horizontally through the air. Stability becomes impossible if the plane is rising at a greater angle than $\tan^{-1}(2 \tan a)$.

68. Now taking the exact condition for the case of horizontal flight

$$\frac{W}{Kg} > \frac{k^2S(1 + \sin^2 a)^2}{e \sin a \cos a} \quad . \quad . \quad . \quad . \quad (122a)$$

and supposing W and S are kept constant, there is an inferior limit to sin a, and consequently a superior limit to the velocity U consistent with stability; that is to say, it is impossible to increase the speed of propulsion of a single plane above a certain limit without the plane becoming unstable. There will also be a maximum value of a consistent with (122a) corresponding to a minimum velocity. The left hand side of (122a) will be found to be a minimum when $a = 28°$ roughly.

On the other hand, in the case of a system of *narrow planes* the condition of stability when expressed in the form (54a) of (§ 48) gives a limiting value for W which is practically independent of the angles of attack of the planes and depends only on their areas and their distances in front of and behind the centre of gravity. *This condition is thus independent of the speed.*

For these reasons it is highly undesirable that the stability of a flying machine should be made to depend on the shifting of the centre of pressure. The machine should be constructed to satisfy the conditions of stability based on the theory of narrow planes discussed in this treatise. The effect of the shift of the centre of pressure is certainly to increase the stability in certain cases, but the higher the velocity the less is the advantage, as may easily be seen for the case of a single lifting machine from equation (118).

Furthermore, the fact that no account has been taken of "rotary derivatives"—the effects of which may be comparable with those of the shifting of the centre of pressure—renders it still more necessary to rely for stability on the separation and difference of inclination of the front and tail planes.

It is important to point out that *these conclusions are not based on the assumption that the rate of shift of the centre of pressure* $a\phi'(\alpha)$ *becomes small when* α *or* μ *is small*, for according to our assumption it tends to the limiting value $-e$.

For large values of α the investigation will require corrections owing to the deviation of the pressure on the plane from the simple "sine law."

Effects of friction — Raised planes. Deviations from the "sine" law.

69. Suppose that, owing to frictional resistance, the resultant thrust on a plane makes a small angle ϵ with the normal. Then a difference will exist between the angle of attack α and the angle α' or $\alpha+\epsilon$ which the resultant thrust makes with the axis of y. The case of ϵ variable is practically covered by the corrections of § 70 below. We shall write in this case $S'=S\cos\alpha\cos\alpha'$ instead of $S\cos^2\alpha$, and $\mu=\tan\alpha$, $\mu'=\tan\alpha'$.

Further, instead of assuming the principle of independence of height we may take account of the height of the planes by writing $x'=x-y\mu'$, $x''=x-2y\mu$; x' will then be the abscissa of the point where the resultant thrust meets the axis of x. If we then write $2\mu-\mu'=\mu''$, so that $x''=x'-y\mu''$, the determinantal equation takes the form

$$\begin{vmatrix} 2\Sigma S'\mu\mu' + \dfrac{W\lambda}{KUg}, & \Sigma S'\mu, & \Sigma S'\mu(x'-y\mu'') - \dfrac{W\cos\theta}{KU\lambda} \\ 2\Sigma S'\mu', & \Sigma S' + \dfrac{W\lambda}{KUg}, & \Sigma S'(x'-y\mu'') + \dfrac{W}{Kg} + \dfrac{W\sin\theta}{KU\lambda} \\ 2\Sigma S'\mu'x', & \Sigma S'x', & \Sigma S'x'(x'-y\mu'') + \dfrac{Wk^2\lambda}{KUg} \end{vmatrix} = 0 \quad . \quad . \quad . \quad (123)$$

If y is the same for both planes it is obvious that the system may be replaced by an equivalent single lifting system with neutral tail raised at the same height y above

the centre of gravity, and the calculations might thus be used to investigate the effects on longitudinal stability of a low centre of gravity. The formation of the coefficients of the biquadratic and substitution in the approximate form of the discriminantal condition of stability presents no special difficulty even in the most general case, but it scarcely appeared desirable to work out the formulæ in this memoir. If the results are wanted, this can always be done subsequently.

70. If the "sine" law of resistance be not assumed, we may still write

$$R = KSU^2 \sin a,$$

where K is now a function of a, say $K(a)$, and according to experimental evidence

$$K(0) = 2K\left(\frac{\pi}{2}\right).$$

In this case the resistance derivatives only involve the values of R and dR/da, and we have

$$\frac{dR}{da} = SU^2\left(K \cos a + \frac{dK}{da} \sin a\right) \quad . \quad . \quad . \quad (124)$$

If we give K the proper value for the value of a under consideration, the whole of the previous work holds good subject to the assumption that dK/da *sin a is negligible in comparison with* K cos a. If not, corrections may be applied. For the single lifting system, these corrections will only occur in X_v, Y_v, where S'_1 will be replaced by

$$S'_1\left(1 + \frac{\mu}{K}\frac{dK}{da}\right)$$

Principal effect of curvature of main surfaces (Camber).

71. From § 32 it easily follows that for a concave surface with neutral tail the variations in the direction of R contribute to X_v an additional term, which is equal to

$$- KS_1 U a/c\ \phi'(a) \sin a \cos a'.$$

If we examine the correction due to this cause alone, assuming the other derivatives to have the same values as in § 48, we find that the correction does not affect $\mathfrak{A}$, $\mathfrak{B}$, or $\mathfrak{E}$; it introduces only terms of order $\sin^2 a$ into $\mathfrak{C}$, while for $\mathfrak{D}$ we now find

$$\frac{\mathfrak{D}}{g^3 U^3} = \frac{2W}{g} K^2 S_1 S_2 l \sin a \sin a' + \underline{2K^3 S_1{}^2 S_2 l^2 \sin^2 a \cos^2 a' \frac{a}{c} \phi'(a)} \qquad (125)$$

If, as is usual with plane surfaces, the centre of pressure shifts forward as the angle of attack decreases, $\phi'(a)$ is negative, and the effect of concavity is to decrease $\mathfrak{D}$, and thus to decrease stability. If, for very small angles of attack, however, the centre of pressure recedes as the angle of attack decreases, the effect of concavity then increases stability. In any case, the effect depends on the term $-X_u Y_v N_r$ in $\mathfrak{D}$ (assuming $N_u = 0$). In view of the difference in the resultant pressure on plane and curved surfaces, it will be more exact if we assume S_1 to be the area of the *equivalent* plane surface.

Effects of "wash."

72. Another case in which the coefficients are modified in a similar way is where the wash produced by the front plane alters the direction in which the air impinges on the rear plane. The angle of attack on the rear plane will be less than the angle which that plane makes with the line of flight, so that the resistance derivatives of the rear plane will again involve two angles a, a'.

In Lanchester's investigation, allowance is further made for the fact that oscillations of the front plane may produce fluctuations in the direction of the wash which affect the pressure on the tail plane. In connection with the present investigation, the method of applying the latter correction for a single lifting system would be as follows:—Let the machine receive a small velocity v perpendicular to the

line of flight. This will produce an additional component velocity of the "wash" current in the same direction which will everywhere be proportional to v when v is small. If the value of this component at the tail be $(1-\epsilon)v$, the effective change in the corresponding relative velocity of the wind on the tail will be ϵv instead of v, consequently the portions of X_v, Y_v, N_v due to the tail will have to be multiplied by ϵ.

In this case the correction is very easily applied. Considering the "simplest case," the determinantal biquadratic, after its last line has been multiplied by $1/\epsilon$, becomes

$$\begin{vmatrix} 2S'_1\mu^2 + \frac{W\lambda}{KUg}, & S'_1\mu, & -\frac{W}{KU\lambda}\cos\theta \\ 2S'_1\mu, & S'_1 + S_2\epsilon + \frac{W\lambda}{KUg}, & -S_2 l + \frac{W}{Kg} + \frac{W\sin\theta}{KU\lambda} \\ 0, & -S_2 l, & S_2\frac{l^2}{\epsilon} + \frac{W\lambda}{KUg}\frac{k^2}{\epsilon} \end{vmatrix} = 0 \tag{126}$$

This equation is deducible from the original one by substituting $S_2\epsilon$ for S_2, l/ϵ for l, and k^2/ϵ for k^2, and we only have to make the corresponding substitutions in the final result, which for horizontal flight now gives

$$U^2\mu > \frac{k^2 g}{l}\frac{S'_1\epsilon + S_2}{S_2} \quad . \quad . \quad . \quad . \quad . \tag{127}$$

agreeing with Lanchester's result. It should be observed that in this case we assume the tail plane to be placed neutrally with respect to the wind. It will therefore make a small angle a_2 with the line of flight, so that the first line of the above determinant ought to read

$$2S'_1\mu^2 + \frac{W\lambda}{KUg}, \quad S'_1\mu + S_2\epsilon\sin a_2 \quad -\frac{W}{KU\lambda}\cos\theta - S_2 l\sin a_2,$$

even if $\cos a_2$ is taken to be unity in the second and third lines. This correction only introduces a small additional term,

$$-2k^2 S'_1\mu S_2\epsilon\sin a_2 \text{ in } \frac{\mathfrak{C}}{W^3}\left(\frac{W}{KUg}\right)^2,$$

and this is usually negligible.

Effect of "lag."—The fluctuations in wash probably do not affect the tail plane at exactly the instant when they are set up by the main plane, but it would be rather premature and unnecessary to investigate the correction for the "lag" thus produced in the present state of our knowledge. It is obvious that the greater the speed of the aeroplane, the less will this correction be. Its existence should, however, be admitted.

Three plane systems.

73. We have seen that the loss of stability which occurs when an aeroplane rises at more than a certain angle is due to $\mathfrak{D}$ decreasing and changing sign. Now $\mathfrak{D}/g^3$ contains the determinant Δ_0 of the resistance coefficients, which for a two plane machine has been proved to be zero. If, however, for *two* we substitute *three* planes or sets of superposed planes at different

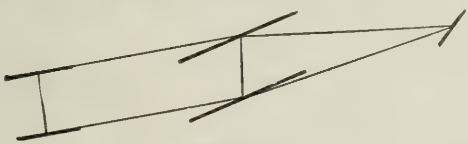

FIG. 28.

horizontal distances from the centre of gravity, Δ_0 may have a value different from zero, and this fact may be used to increase the stability. For example, the machine may be provided with a tail plane in the rear, and a rudder plane or elevator in front of the main-supporting surfaces, as in the Farman and Curtiss biplanes (Fig. 28).

Let S_1, S_2, S_3 be the areas, x_1, x_2, x_3 the abscissæ or horizontal co-ordinates, α_1, α_2, α_3 the angles of attack of the three planes, then with the usual notation $S'_1 = S_1 \cos^2 \alpha_1$, $\mu_1 = \tan \alpha_1$, etc., we have

$$\frac{\Delta_0}{K^3 U^3} = 2 \begin{vmatrix} \Sigma S'\mu^2 & \Sigma S'\mu & \Sigma S'\mu x \\ \Sigma S'\mu & \Sigma S' & \Sigma S'x \\ \Sigma S'\mu x & \Sigma S'x & \Sigma S'x^2 \end{vmatrix}.$$

By the rule for multiplication of two determinants this is equal to

$$2\begin{vmatrix} S'_1\mu_1, & S'_2\mu_2, & S'_3\mu_3 \\ S'_1, & S'_2, & S'_3 \\ S'_1x_1, & S'_2x_2, & S'_3x_3 \end{vmatrix} \times \begin{vmatrix} \mu_1, & \mu_2, & \mu_3 \\ 1, & 1, & 1 \\ x_1, & x_2, & x_3 \end{vmatrix} = 2S'_1S'_2S'_3 \begin{vmatrix} \mu_1, & \mu_2, & \mu_3 \\ 1, & 1, & 1 \\ x_1, & x_2, & x_3 \end{vmatrix}^2$$

$$= 2S'_1S'_2S'_3\{(\mu_1 - \mu_3)(x_2 - x_3) - (\mu_2 - \mu_3)(x_1 - x_3)\}^2 \quad . \quad (128)$$

and is essentially positive, unless it be zero.

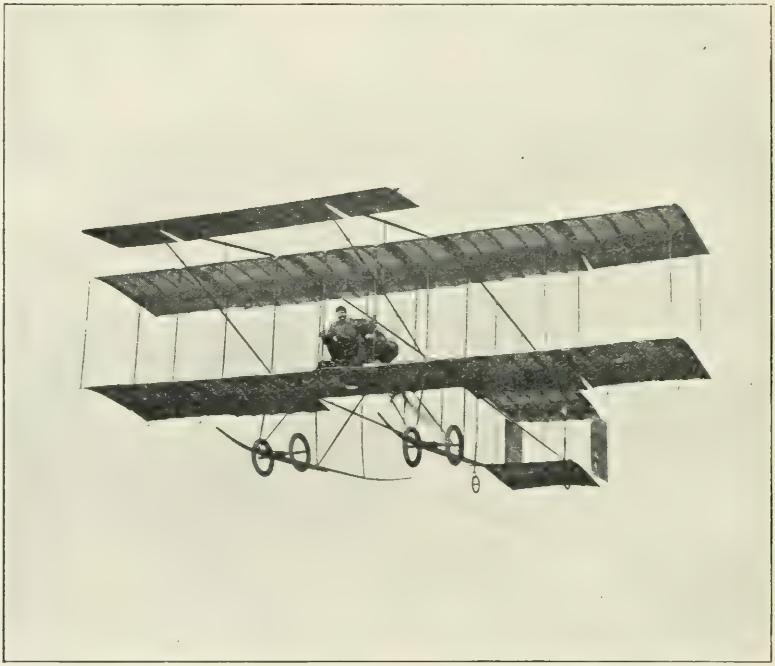

Photo.] [*The Sport and General Illustrations Co.*

FIG. IV.—M. HENRY FARMAN FLYING AT BLACKPOOL.

An aeroplane with auxiliary surfaces fore and aft constituting a "three plane" system as in § 73, each pair of superposed planes counting as a single plane for the purposes of longitudinal stability. The conditions of *lateral stability* are not all satisfied. The vertical rudder planes in the rear are equivalent to a single vertical fin behind and not much above the centre of gravity, so that the coefficient $\mathfrak{E}_1$ is negative (indicating instability), though $\mathfrak{D}_1$ and $\mathfrak{H}_1$ are positive.

The effect of this term is thus to increase, never to decrease, symmetric stability.

It remains to examine the effect on the other coefficients in the biquadratic, and for this purpose we

shall again introduce the notion of an *equivalent system of planes.*

We can, as in § 59, choose a single lifting surface with neutral tail having the values of $\Sigma S'\mu^2$, $\Sigma S'\mu$, $\Sigma S'$, $\Sigma S'x\mu$, $\Sigma S'x$, the same as for the given three plane system, but in view of the fact that Δ_0 is not zero, the value of $\Sigma S'x^2$ will no longer be the same, that is, it will no longer be equal to S_2l^2 in the second system.

To get over this difficulty we imagine the neutral plane divided into two parts, say equal parts for simplicity, and

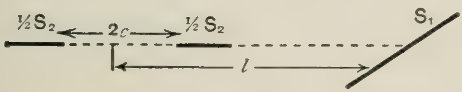

FIG. 29.

these parts separated by a distance $2c$, the middle point between them still being at a distance l behind the centre of gravity. The new system is thus specified as follows (Fig. 29):

Areas of planes	S_1	$\frac{1}{2}S_2$	$\frac{1}{2}S_2$
Inclinations	a	0	0
Co-ordinates	0	$-l+c$	$-l-c.$

The value of $\Sigma S'x^2$ will then be equal to $S_2(l^2+c^2)$, the relations between the original and the new system being as follows:

Original System.	Equivalent System.
$\Sigma S \sin^2 a$	$S_1 \sin^2 a$
$\Sigma S \sin a \cos a$	$S_1 \sin a \cos a$
$\Sigma S \cos^2 a$	$S_1 \cos^2 a + S_2$
ΣS	$S_1 + S_2$
$\Sigma Sx \sin a \cos a\ (=0)$	0
$\Sigma Sx \cos^2 a$	$-S_2l$
$\Sigma Sx^2 \cos^2 a$	$+S_2(l^2+c^2)$ (129)

For the equivalent system we easily find

$$\frac{\Delta}{K^3U^3} = 2S_1S_2{}^2c^2\sin^2 a \quad . \quad . \quad . \quad . \quad . \quad (130)$$

and since Δ_0 for the original system has been shown to be essentially positive, it follows that c^2 is positive, so that the distance of separation of the two parts of the neutral plane is real; in other words, the equivalent system is a possible one.

74. We now form the coefficients of the biquadratic for the equivalent system, taking into account inclination of flight path but not head resistance, always supposing the propeller thrust to act along the axis of x and to be constant. To the usual first approximation

$$\mathfrak{A} = CW^2, \quad \frac{\mathfrak{B}}{gUK} = CW(S'_1 + S_2) + W^2S_2(l^2 + c^2) \quad . \quad (131a, b)$$

$$\frac{\mathfrak{C}}{g^2U^2K^2} = WS'_1S_2(l^2 + c^2) + \frac{W^2}{Kg}S_2l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131c)$$

$$\frac{\mathfrak{D}}{g^3U^3K^3} = \underline{2S'_1S_2{}^2c^2\mu^2} + \frac{W}{gK}S'_1S_2l\mu(2\mu + \tan\theta) \quad . \quad . \quad . \quad (131d)$$

$$\frac{\mathfrak{E}}{g^4U^4K^4} = \frac{2W}{U^2K^2g}S'_1S_2l\mu(\cos\theta + \mu\sin\theta) = \frac{2}{Kg}S'_1{}^2S_2l\mu^2(1 + \mu\tan\theta)$$

$$= \frac{2}{Kg}S'_1{}^2S_2\mu^2 l \text{ to first order} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131e)$$

If we suppose $\theta = 0$ in the first instance and form the approximate condition $\mathfrak{C}\mathfrak{D} - \mathfrak{E}\mathfrak{B} > 0$, we shall find that the only parts containing c^2 which do not cancel are those arising from the product of $\mathfrak{C}$ into the term underlined in $\mathfrak{D}$, which arises from the determinant Δ_0, and the condition of stability becomes

$$\frac{W}{Kg}S_2l - k^2(S'_1 + S_2)S'_1$$
$$+ \left\{\frac{\tan\theta}{2\mu} + \frac{Kg}{W}\frac{S_2c^2}{l}\right\}\left\{\frac{W}{Kg}S_2l + S'_1S_2(l^2 + c^2)\right\} > 0 \quad . \quad . \quad (132)$$

The approximate condition for stability is the same as it would be in horizontal flight with $c = 0$, when the direction

of flight makes an upward angle with the horizon given by

$$\tan\theta = -2\frac{Kg}{W}\frac{S_2c^2\mu}{l} = -2\frac{S_2}{S_1}\frac{g\cos\theta}{U^2}\frac{c^2}{l} \quad . \quad . \quad (133)$$

and this represents the increase in the angle of elevation consistent with stability obtainable by separating the halves of the neutral plane through a distance $2c$.

To sum up, then, the values of $\mathfrak{B}$ *and* $\mathfrak{C}$ *are increased by the substitution of* $l^2 + c^2$ *for* l^2. *The value* $\mathfrak{D}$ *is increased by the additional term due to the determinant* Δ_0, *though for ascending flight it is decreased by the term involving* $\tan\theta$, *which is then negative. The net result is an increase in the approximate discriminant* $\mathfrak{C}\mathfrak{D} - \mathfrak{E}\mathfrak{B}$, *this increase being equal to the increase in* $\mathfrak{D}$ *multiplied by the increased value of* $\mathfrak{C}$. For the original system the increased elevation obtainable by this means is given approximately by the relation

$$\Delta_0 - W^2/g\, N_v \sin\theta = 0 \quad (N_v \text{ negative, } \theta \text{ negative}) \quad . \quad (133a)$$

as is evident from equations (18a).

The question of the hitherto neglected rotary coefficients of broad planes.

75. The transformations and methods of the last article help us to obtain some estimate of the general effects of the rotary coefficients $f_r(a)$ and $\phi_r(a)$, if only as deductions from assumed hypotheses concerning them.

A plausible assumption is that when a plane rotates *about its centre of pressure*, the resultant thrust is not affected by the rotation to any important extent—possibly not at all, but that the *moment* of the pressures so set up tends to retard rotation. If this be assumed, the effect is represented by an additional positive term in N_r, and can be made equivalent to the effect, considered in the last article, of changing S_2l^2 into $S_2(l^2 + c^2)$ by a suitable choice

of c. The effect is thus to increase stability and in particular to increase the angle of elevation of the flight path consistent with stability, as indeed is almost evident from general considerations.

If rotation about the centre decreases the moment of the resistances, the effect will be to decrease stability. This alternative hypothesis seems much less plausible than the first, but questions of the kind cannot be decided by rule-of-thumb reasoning. In this connection, it is a significant fact that a balanced plane which is free to rotate on its axis will rotate continuously when placed in a wind, but the decisive test is obtainable (§ 29) by attaching the plane to a pendulum placed in an air current, the axis of suspension passing through the centre of pressure, and observing whether the oscillations subside or increase.

CHAPTER VII.

ASYMMETRIC OR "LATERAL" STABILITY—STRAIGHT PLANES AND VERTICAL FINS.

Further hypotheses regarding narrow planes.

76. In dealing with the longitudinal or symmetric oscillations, we assumed as a definition of *narrow* planes, that the chord or distance from the front to the back edge of a plane was so small that the unequal velocities of the two edges due to rotation had a negligible effect on the distribution of pressure. On the other hand, in studying

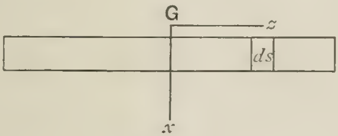

FIG. 30.

the asymmetric oscillations we have to take account of the fact that the *span* of the planes is often considerable, so that rotation may produce considerable differences in the velocity at different distances from the axis of x, or rather from the plane of symmetry.

Let the plane be divided into strips such as dS in the accompanying figure. Then the new assumption which we have to make, and which we agree to regard as implied in the definition of a *narrow plane*, is that the pressure on the element dS is independent of the motion of parts

to the right and left of the element, and depends only on the velocity of the element itself. A further assumption which we regard as implied in the statement that *the angle of attack is small* is that the pressure on dS follows the well-known sine law, or, what is equivalent, that the resultant thrust on $dS = KdS \times$ resultant velocity $\times$ normal velocity relative to the air.

A correction is probably necessary, at least near the extremities of the plane, but it is no use trying to take all such corrections into account at the outset of the discussion (see § 93).

To make the argument more general, and applicable in particular to surfaces which are bent up or down towards

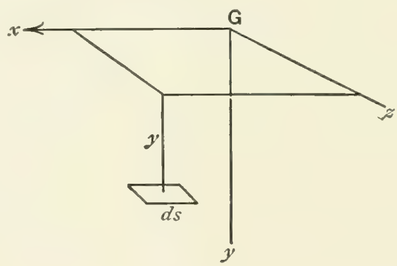

Fig. 31.

their extremities, let us suppose that x, y, z are the co-ordinates of the element dS, and that l, m, n are the direction cosines of the normal to this element (Fig. 31).

The resultant velocity of dS is

$$U + u - yr + zq,$$

neglecting terms of the second order, while the normal velocity is

$$l(U + u - yr + zq) + m(v - zp + xr) + n(w - xq + yp),$$

so that, again to the first order, we have for the resultant thrust dR on the element

$$dR = KdS\{lU^2 + 2lU(u - yr + zq) + mU(v - zp + xr) + nU(w - xq + yp)\} \quad (134)$$

and the corresponding force and couple components, when integrated over all the elements, become

$$X = \int l dR, \qquad Y = \int m dR, \qquad Z = \int n dR,$$
$$L = \int (ny - mz) dR, \; M = \int (lz - nx) dR, \; N = \int (mx - ly) dR. \quad (135)$$

If the aeroplane is symmetrical about the plane of x, y, such integrals as contain only odd powers of z or of n (when dR is expressed in terms of dS by means of 134) will vanish, and this will secure the conditions that the resistance derivatives $(X, Y, N)_{w, p, q}$ and $(Z, L, M)_{u, v, r}$ vanish, *i.e.* the independence of the symmetric and asymmetric oscillations. We obtain

$$X = U^2 \int K l^2 dS + 2Uu \int K l^2 dS + Uv \int K lm dS$$
$$+ Ur \int K l(mx - 2ly) dS \quad . \quad . \quad (136x)$$

$$Y = U^2 \int K lm dS + 2Uu \int K lm dS + Uv \int K m^2 dS$$
$$+ Ur \int K m(mx - 2ly) dS \quad . \quad . \quad (136y)$$

$$N = U^2 \int K l(mx - ly) dS + 2Uu \int K l(mx - ly) dS + Uv \int K m(mx - ly) dS$$
$$+ Ur \int K (mx - ly)(mx - 2ly) dS \quad (136n)$$

$$Z = Uw \int K n^2 dS + Up \int K n(ny - mz) dS$$
$$+ Uq \int K n(2lz - nx) dS \quad . \quad (136z)$$

$$L = Uw \int K n(ny - mz) dS + Up \int K (ny - mz)^2 dS$$
$$+ Uq \int K (ny - mz)(2lz - nx) dS \quad (136l)$$

$$M = Uw \int K n(lz - nx) dS + Up \int K (ny - mz)(lz - nx) dS$$
$$+ Uq \int K (lz - nx)(2lz - nx) dS \quad (136m)$$

The first three refer to longitudinal stability. By putting $l = \sin a$, $m = \cos a \cos \beta$, $n = \cos a \sin \beta$ they might be applied to the case of an aeroplane whose surfaces slope upwards at an angle β towards their extremities. This case reduces to the problems considered in Chapters V, VI by putting $\cos \beta = 1$. In most cases of immediate interest, this approximation will be amply sufficient, and

therefore it is considered undesirable to reproduce a fuller discussion here. In lateral stability the difference is important and we start with the simplest case.

Straight planes.

77. By this we mean planes perpendicular to the plane of x, y, in contradistinction to the bent up planes just mentioned. For these we put $l = \sin \alpha$, $m = \cos \alpha$, $n = 0$, and obtain

$$Z_w, Z_p, Z_q, L_w, M_w \text{ each } = 0$$

and writing

$I = \int z^2 dS =$ moment of inertia of area of plane with respect to the plane of x, y, we have

$$\begin{aligned} L_p &= KUI \cos^2 a & L_q &= -2KUI \sin a \cos a \\ M_p &= -KUI \sin a \cos a, & M_q &= +2KUI \sin^2 a \end{aligned} \quad . \quad . \quad (137)$$

Substituting in the expressions for the coefficients in § 20, we find $\mathfrak{D}$ and $\mathfrak{E}$ (omitting the suffixes) both zero, also for a single plane $\mathfrak{E} = 0$. If there are two planes whose angles of attack are α_1, α_2, and moments of inertia I_1 and I_2 it will be found that

$$\frac{\mathfrak{E}}{K^2 U^2 g^2} = 2WI_1 I_2 \sin^2(a_1 - a_2) \quad . \quad . \quad . \quad . \quad . \quad (138)$$

which also vanishes if $\alpha_1 = \alpha_2$ or the planes are parallel, likewise if I_1 or $I_2 = 0$ or one of the planes is a rudder plane of negligible span.

In every case two, if not three, roots of the biquadratic vanish. This indicates lack of stability, the effects of which can be studied by writing down the equations of motion

$$W\left(\frac{dw}{gdt} - \frac{qU}{g}\right) = -W \sin \phi \cos \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad (139w)$$

$$A \frac{dp}{gdt} - F \frac{dq}{gdt} = KUI(-p \cos^2 a + 2q \sin a \cos a) \quad . \quad (139p)$$

$$B \frac{dq}{gdt} - F \frac{dp}{gdt} = KUI(+p \sin a \cos a - 2q \sin^2 a) \quad . \quad (139q)$$

The last two show that there is no tendency for the machine to right itself if it heels on one side, for there is no term in them depending on the action of gravity, and therefore no tendency to assume a definite position relative to the vertical. The first tells us that if the machine has heeled over through an angle ϕ, gravity will tend to make it slide down sideways, and there is nothing to check this tendency. The maintenance of lateral balance must therefore be effected by furnishing the machine with warping devices to be controlled by the operator, or else other planes, such as vertical fins or rudders, must be added.

The explanation of the third vanishing root of the biquadratic in the case of a single plane may be seen by multiplying the second and third equations above (139 p, q) by sin α and cos α, and adding; this eliminates the right hand side, and the resulting equation expresses the fact that the couple about a normal to the plane is zero, so that there is nothing tending to check rotation about this normal, *i.e.* there is no directional stability.

The following points should be noticed.

(1) *The effect of a straight narrow plane on the lateral stability depends on the moment of inertia of its area about its plane of symmetry, and does not depend on the position of the plane.* For x and y do not enter into the coefficients.

(2) *A horizontal rudder plane for steering in a vertical plane does not affect the stability unless its span is considerable.*

(3) We may write $I=SR^2$ where S is the area and R the radius of gyration of the plane. We may, following Lanchester, extend the method to cases other than those of *narrow* straight planes by suitable conventions as to the value of R in the various terms, these becoming his "aerodynamic" and "aerodromic radii."

Effect of a single vertical fin.

78. If we suppose a vertical fin or plane of area T placed in or parallel to the plane of x, y with its centre of pressure (for grazing incidence) at the point $(x, y, 0)$ or (x, y, z), the resistance derivatives due to this fin are given by

$$\begin{array}{lll} Z_w = K'TU & Z_p = K'TUy & Z_q = -K'TUx \\ L_w = K'TUy & L_p = K'TUy^2 & L_q = -K'TUxy \\ M_w = -K'TUx & M_p = -K'TUxy & M_q = K'TUx^2 \end{array} \quad . \quad (140)$$

while for a system of several such fins we only have to prefix the sign of summation to expressions of these forms. These expressions are, of course, to be added to the corresponding derivatives due to the main planes. In the case of a single main plane and a single fin we obtain

$$\frac{\mathfrak{D}}{U^3g^3} = KK'IT\frac{W}{g}(y\sin a - x\cos a)\cos a$$
$$- \frac{W}{U^2g}K'T\{(By - Fx)\cos\theta + (Ax - Fy)\sin\theta\} \quad . \quad (141d)$$
$$\frac{\mathfrak{E}}{U^4g^4} = \frac{W}{gU^2}KK'TI(2\sin a\cos\theta - \cos a\sin\theta)(x\cos a - y\sin a) \quad (141e)$$

Consider the case of horizontal flight $(\theta = 0)$, and, to simplify matters, suppose the axis of x is a principal axis of inertia, so that $F = 0$, an assumption we shall frequently find it convenient to make. In this case, the second term in the expression giving $\mathfrak{D}$ reduces to the part containing By.

Now the only way of securing the stability condition $\mathfrak{E}$ positive is by making $x\cos a - y\sin a$ positive, that is, placing the fin or rudder *in front* of a line through the centre of gravity perpendicular to the main plane; *i.e.* briefly speaking, placing the rudder in front. But this necessarily makes the first term of $\mathfrak{D}$ negative, a result which can only be counteracted by making y negative and sufficiently large in the second term of $\mathfrak{D}$, that is, by

raising the fin considerably above the centre of gravity. Mr. Harper and I have investigated the matter at some length, and find that even when stability could theoretically be secured by this means the height of the fin would have to be far too large, in proportion to its horizontal co-ordinate x, to enable the method to be applied in the construction of a convenient flying machine, and further that the difficulty increases with the velocity U. There is, of course, a possibility of improving matters by making the product of inertia F positive, but this again does not appear very promising.

In a flying machine of the now prevalent types, y is as a rule not very different from zero, and the choice lies between putting the vertical rudder behind and making the machine unstable through $\mathfrak{E}$ negative or putting it in front and making it unstable through $\mathfrak{D}$ negative. For a machine, the dimensions of which were given, it would be easy to obtain in either case numerical approximate solutions of the biquadratic and to decide which was the lesser of the two evils, but a more interesting line of investigation is to examine how two fins are better than one in securing stability.

General case of a number of fins.

79. It will be observed that (1) the resistance derivatives due to vertical fins parallel to the plane of x, y are independent of their z co-ordinates; thus, for example, two vertical planes symmetrically placed at opposite sides of the plane of x, y are equivalent to a single plane of double the area in the plane of x, y midway between them.

(2) For a number of fins the resistance derivatives contain the sums of first and second powers of their co-ordinates, and are thus transformable by the rules

applicable to centres of gravity and moments of inertia (compare Lanchester's "Fin Resolution").

Writing T for the *sum* of the areas of the fins, x, y for the co-ordinates of their centre of mean position (or centre of pressure), and M_1, M_2, P for the moments and product of inertia of the areas of the fins with respect to axes parallel to the co-ordinate axes through the latter

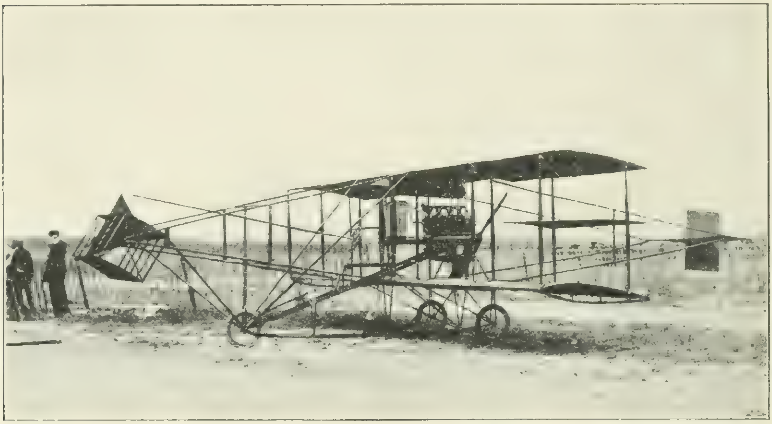

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FIG. V.—GLEN CURTISS BIPLANE AT RHEIMS.

An aeroplane with two vertical fins, namely a triangular fin in front and a rectangular rudder plane behind. This arrangement is favourable to lateral stability if the centre of pressure of the two fins is slightly above or in front of the centre of gravity and certain other conditions are satisfied (§§ 84–86).

point, the expressions Z_w, Z_p, Z_q, L_w, M_w of the last article hold good, but we have

$$L_p = K'U(Ty^2 + M_1),\ L_q = M_p = -K'U(Txy + P),\ M_q = K'U(Tx^2 + M_2) \tag{142}$$

To shorten the algebra assume provisionally that $K' = K$. If the fins have a different resistance-coefficient K' from the main planes we can subsequently restore the difference by putting a factor K'/K into the terms depending on the areas and moments of the fins.

The determinantal equation is most conveniently now put in the form of an equation in λ/KUg, and becomes

$$\begin{vmatrix} W\dfrac{\lambda}{KUg} + T, & \dfrac{W\cos\theta}{K^2U^2g}\left(\dfrac{KUg}{\lambda}\right) + Ty, & -\dfrac{W}{Kg} - \dfrac{W\sin\theta}{K^2U^2g}\left(\dfrac{KUg}{\lambda}\right) - Tx \\ Ty, & A\dfrac{\lambda}{KUg} + Ty^2 + M_1 + I\cos^2 a, & -F\dfrac{\lambda}{KUg} - (Txy + P) - 2I\sin a\cos a \\ -Tx, & -F\dfrac{\lambda}{KUg} - (Txy + P) - I\sin a\cos a, & B\dfrac{\lambda}{KUg} + (Tx^2 + M_2) + 2I\sin^2 a \end{vmatrix} = 0 \quad . \quad . \quad . \quad (143)$$

At this stage, it will be found convenient to write I' for $I\cos^2 a$ and μ for $\tan a$, consistently with the practice previously adopted in dealing with longitudinal stability.

When the determinant is developed into the biquadratic form, the following expressions represent the coefficients if λ/KUg is taken as variable, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc., representing the coefficients if λ is taken as variable :—

$$\mathfrak{A} = W(AB - F^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144a)$$

$$\begin{aligned}\frac{\mathfrak{B}}{KUg} = {} & T(AB - F^2) \\ & + W[A(2I'\mu^2 + Tx^2 + M_2) \\ & - F(3I'\mu + 2Txy + 2P) + B(I' + Ty^2 + M_1] \quad . \quad . \quad (144b)\end{aligned}$$

$$\begin{aligned}\frac{\mathfrak{C}}{K^2U^2g^2} = {} & T[A(2I'\mu^2 + M_2) - F(3I'\mu + 2P) + B(I' + M_1)] \\ & + WI'[2(Ty^2 + M_1)\mu^2 - 3(Txy + P)\mu + (Tx^2 + M_2)] \\ & + WT(M_1x^2 - 2Pxy + M_2y^2) + W(M_1M_2 - P^2) \\ & - \frac{WT}{Kg}(Ax - Fy) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144c)\end{aligned}$$

$$\begin{aligned}\frac{\mathfrak{D}}{K^3U^3g^3} = {} & TI'[2M_1\mu^2 - 3P\mu + M_2] + T(M_1M_2 - P^2) \\ & - \frac{WT}{Kg}[(M_1 + I')x - (P + I'\mu)y] \\ & - \frac{TW\cos\theta}{K^2U^2g}(By - Fx) - \frac{TW\sin\theta}{K^2U^2g}(Ax - Fy) \quad . \quad (144d)\end{aligned}$$

$$\begin{aligned}\frac{\mathfrak{E}}{K^4U^4g^4} = {} & \frac{WTI'}{K^2U^2g}(x - y\mu)(2\mu\cos\theta - \sin\theta) \\ & + \frac{WT}{K^2U^2g}\{\cos\theta(Px - M_2y) - \sin\theta(M_1x - Py)\} \quad . \quad (144e)\end{aligned}$$

These expressions become much simplified when (1) the angle of attack a of the main planes is small so that μ is small, (2) the fins are so small that we need only retain the lowest *significant* terms involving T or the corresponding moments, and in particular M_1, M_2, P are small com-

pared with I, (3) the product of inertia F vanishes. (4) $\theta = 0$ or is small. In these cases, we have to a sufficient approximation

$$\mathfrak{A} = WAB, \quad \frac{\mathfrak{B}}{KUg} = WIB \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145a, b)$$

$$\frac{\mathfrak{C}}{K^2U^2g^2} = TIB - \frac{TW}{Kg}Ax + T(AM_2 + BM_1) + WI(Tx^2 + M_2) \qquad (145c)$$

$$\frac{\mathfrak{D}}{K^3U^3g^3} = TM_2I - \frac{WTIx}{Kg} - \frac{TW}{K^2U^2g}By \quad . \quad . \quad . \quad . \quad . \quad (145d)$$

$$\frac{\mathfrak{E}}{K^4U^4g^4} = \frac{WTI}{K^2U^2g}x(2\mu - \tan\theta)\cos\theta - \frac{WT}{K^2U^2g}(M_2y - Px) \quad . \quad (145e)$$

80. From these forms, we notice the following points :—

(1) The moment of inertia M_1 of the fins does not enter into the approximate values except in the part of $\mathfrak{C}$ which is of order T^2, although the height y of their centre does enter. There is thus no advantage in making M_1 different from zero, that is, placing two fins one above the other.

(2) It is easy to see that $M_1M_2 - P^2$ is zero for two fins, and positive for any number greater than two unless collinear. As this expression occurs among the neglected terms, there is no advantage in using more than two fins.

General character of the oscillations—Approximate resolution of the biquadratic—Resistance to rolling.

81. The conclusions obtained in connection with the separation of long and short oscillations in the case of longitudinal stability do not hold good in the case of lateral or asymmetric oscillations, and in the present case the separation of the roots of the biquadratic assumes a different form.

In the first place, if there are no fins, only the first two terms of the biquadratic remain finite, and these give rise

to one finite value of λ, say λ_1, which $= -\mathfrak{B}/\mathfrak{A}$. With the approximate values of (145 a, b) we have

$$\frac{\lambda_1}{KUg} = -\frac{I'}{A} \quad . \quad . \quad . \quad . \quad . \qquad (146)$$

whence λ_1 is essentially negative as required for stability. This solution determines the rate of damping of an angular velocity p about the axis of x, as is evident from first principles. It is, in fact, equivalent to the dynamical equation,

$$A\frac{dp}{gdt} = -KUI'p \quad . \quad . \quad . \quad . \quad . \qquad (146a)$$

While this damping tends to check the rolling motion about the line of flight, it indicates no tendency to restore equilibrium when the machine has heeled over through a definite angle about this line.

Even when F is not zero and a not small, λ_1 will in general be negative. For $AB - F^2$ is essentially positive by the theory of moments of inertia, so that $\mathfrak{A}$ is positive, and writing $A = \Sigma m(y^2 + z^2)$, etc., as in elementary Rigid Dynamics, we find (with T, M_1, M_2 and P each zero)

$$\frac{\mathfrak{B}}{KUg} = WI\left\{\Sigma m(x \cos a - y \sin a)(x \cos a - 2y \sin a) + 3\Sigma mz^2\right\} \quad . \qquad (147)$$

an expression which could only be negative in the case of a preponderance of the mass of the flying machine between the planes $x \cos a - y \sin a = 0$ and $x \cos a - 2y \sin a = 0$. A machine the mass of which was so distributed might be expected to exhibit anomalous behaviour, but would not occur ordinarily in practice.

If the fin area T instead of being zero is small, the remaining roots λ_2, λ_3, λ_4, of the biquadratic will be small, compared with λ_1. They will, therefore, all make $\mathfrak{A}\lambda^4$ small compared with $\mathfrak{B}\lambda^3$, and therefore will be given to a first approximation by the cubic

$$\mathfrak{B}\lambda^3 + \mathfrak{C}\lambda^2 + \mathfrak{D}\lambda + \mathfrak{E} = 0 \quad . \quad . \quad . \quad . \qquad (148)$$

the condition of stability assuming the simpler approximate form

$$\mathfrak{C}\mathfrak{D} - \mathfrak{E}\mathfrak{B} > 0,$$

which greatly simplifies some of the calculations, and will be used hereafter when the full condition is too complicated.

82. In certain circumstances the separation can be carried further. We notice that $\mathfrak{C}/K^2U^2g^2$ and $\mathfrak{D}/K^3U^3g^3$ contain terms of the first order in T, while in $\mathfrak{E}/K^4U^4g^4$ one term contains a factor of order Ta, and the other is of the second order in T. As there are two independent small quantities a and T, it is not possible to lay down a universal rule, but we may generally expect the biquadratic to take the form

$$\mathfrak{A}\lambda^4 + \mathfrak{B}\lambda^3 + \mathfrak{C}'\lambda^2 + \mathfrak{D}'\lambda + \mathfrak{E}'' = 0,$$

the first two coefficients $\mathfrak{A}$, $\mathfrak{B}$ being finite $\mathfrak{C}'$, $\mathfrak{D}'$ of the first order, $\mathfrak{E}''$ of the second order. We have discussed the finite root $\lambda_1 = -\mathfrak{B}/\mathfrak{A}$. Another root, λ_4, is given approximately by $\lambda_4 = -\mathfrak{E}''/\mathfrak{D}'$ and is of the first order. The middle pair of roots, λ_2 and λ_3, are of order $\frac{1}{2}$, the first approximation to their values being given by

$$\mathfrak{B}\lambda^2 + \mathfrak{D}' = 0,$$

and using this value we can substitute

$$\mathfrak{A}\lambda^4 = -\frac{\mathfrak{A}\mathfrak{D}'}{\mathfrak{B}}\lambda^2, \quad \mathfrak{E}'' = -\frac{\mathfrak{E}''\mathfrak{B}}{\mathfrak{D}'}\lambda^2,$$

leading to the second approximation given by

$$\mathfrak{B}\lambda^2 + \left\{\mathfrak{C}' - \frac{\mathfrak{A}\mathfrak{D}'}{\mathfrak{B}} - \frac{\mathfrak{E}''\mathfrak{B}}{\mathfrak{D}'}\right\}\lambda + \mathfrak{D}' = 0 \quad . \quad . \quad (149)$$

Omitting the accents, we see that the modulus of decay of the oscillations determined by the middle pair of roots λ_2 and λ_3 is

$$\frac{\mathfrak{B}\mathfrak{C}\mathfrak{D} - \mathfrak{A}\mathfrak{D}^2 - \mathfrak{E}\mathfrak{B}^2}{\mathfrak{B}^2\mathfrak{D}} \quad . \quad . \quad . \quad . \quad (150)$$

thus being directly related to the discriminant $\mathfrak{H}$. The method will fail and all the roots will be of the same order, if $\mathfrak{E}/\mathfrak{D}$ be of the order of smallness of $(\mathfrak{D}\,\mathfrak{B})^{\frac{1}{2}}$ that is if T be of order a^2. If the method is applicable, then the various roots correspond to:—

(1) A rapid subsidence determined by λ_1 (damping of the angular velocity of rolling).

(2) A slow subsidence determined by λ_4.

(3) An oscillation of medium period determined by λ_2 and λ_3, but the rate of subsidence of which may be comparable with that corresponding to λ_4.

83. An important point is carefully to distinguish the terms depending directly on the action of gravity. These are the terms containing $\cos\theta$ and $\sin\theta$ in the full expressions for $\mathfrak{D}/K^3U^3g^3$ and $\mathfrak{E}/K^4U^4g^4$. In the abbreviated or approximate expressions, they are distinguished by having U^2 in the denominator. These are the only terms which determine any tendency on the part of the machine to assume or return to a definite position relative to the vertical. No method which ignores these terms will give inherent stability such as will enable a machine to right itself when displaced. The presence of U^2 in the denominator is in agreement with the fact that while the air pressure on a moving plane increases with the square of the velocity, the effects of gravity remain constant, and, therefore, their *relative* influence decreases.

The occurrence of g elsewhere is due to our having measured resistances in pounds instead of poundals, and our use of the *homogeneous* form of the equations of dynamics.

Case I.—Systems with two raised fins at the same height.

84. We have already pointed out in dealing with one fin that $x\cos a - y\sin a$ occurs with a positive coefficient

in $\mathfrak{C}$ and with a negative coefficient in $\mathfrak{D}$. The simplest way out of this difficulty is to make $x \cos a - y \sin a = 0$, that is, to put the centre of pressure of the fins in a line through the centre of gravity perpendicular to the main plane. Since a is small, it comes practically to the same thing if $x = 0$, *i.e.* if the centre of pressure is vertically above the centre of gravity. This arrangement is mentioned by Lanchester, who refers to its advantages in securing stability. A further simplification consists in assuming the two fins to be on the same level, so that y is

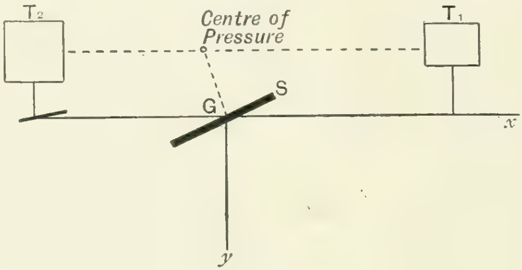

Fig. 32.

the same for both; in this case M_1 and P will vanish, and if T_1, T_2 (Fig. 32) are the areas, x_1, x_2 the co-ordinates of the fins, we shall have

$$M_2 = \frac{T_1 T_2}{T_1 + T_2}(x_1 - x_2)^2$$

$$= \frac{T_1 T_2}{T_1 + T_2} \times \text{square of distance between fins} \qquad (151)$$

The condition $\mathfrak{C}$ positive now reduces to y negative, that is, *for stability the fins must be raised above the centre of gravity.*

The expression for $\mathfrak{D}$ reduces to

$$\frac{\mathfrak{D}}{K^3 U^3 g^3} = TM_2 U - \frac{TW}{K^2 U^2 g}\left\{(By - Fx)\cos\theta + (Ax - Fy)\sin\theta\right\} \quad (152)$$

Remembering that x is small, the condition y negative will certainly make $\mathfrak{D}$ positive if $F = 0$, and $\mathfrak{C}$ will also be necessarily positive under the same conditions. If F

is different from zero, there will still be wide limits consistent with the stability conditions $\mathfrak{D}$ and $\mathfrak{E}$ positive.

85. The final condition $\mathfrak{H}$ or $\mathfrak{B}\mathfrak{C}\mathfrak{D} - \mathfrak{A}\mathfrak{D}^2 - \mathfrak{E}\mathfrak{B}^2 > 0$ necessarily becomes complicated, and it is useful to obtain some simpler condition which will suffice for the purpose. Now, after a good deal of laborious substitution, I find that *when the tangent of the inclination* μ *is small, the condition*

$$(-y) < \frac{K^2U^2gI^2}{WA} \quad . \quad . \quad . \quad . \quad (153)$$

substituted in $\mathfrak{H}$ *makes that expression essentially positive*; this condition is therefore sufficient for stability.

In verifying this conclusion, it will be found best, in the first instance, to make the substitution in $\mathfrak{C}\mathfrak{D} - \mathfrak{E}\mathfrak{B}$, and with the value so obtained to substitute in

$$(\mathfrak{C}\mathfrak{D} - \mathfrak{E}\mathfrak{B})\mathfrak{B} - \mathfrak{A}\mathfrak{D}^2.$$

The work is rather laborious.

For the case where higher powers of μ are retained, I have tried the same assumed condition, which, of course, may be written

$$(-y) < \frac{K^2U^2gI^2\cos^4 a}{WA} \quad . \quad . \quad . \quad (153a)$$

but this substituted in $\mathfrak{H}$ leaves certain negative terms in it. These are *probably* much smaller than the positive parts in cases likely to occur in applications; at the same time their presence shows that the above condition can only be relied on as sufficient provided that a further condition is satisfied. It might perhaps be worth while to examine this particular case at some future time.

The above condition may be put into various modified forms by combining it with the condition of equilibrium $W = KSU^2 \sin a$. If we put $A = Wr_1^2$ and $I = S\rho^2$, then r_1 and ρ will be radii of gyration of the mass of the machine and of the area of its main plane respectively, and we find, on substituting

$$(-y)2H\sin^2 a < \frac{\rho^4}{r_1^2}, \text{ if } H = \frac{U^2}{2g} \quad . \quad . \quad . \quad (153b)$$

H being the height to which the velocity U is due.

Even the above formula is not required if the total area of the fins is small compared with that of the main planes. It will, in fact, be found that the terms of $\mathfrak{H}$, which are of the lowest order in T and M_2, are essentially positive.

It will thus be seen that the present arrangement possesses a wide range of stability. Moreover, this stability is not much affected by the angle θ which the flight path makes with the horizon, $\mathfrak{C}$, and the important parts of $\mathfrak{D}$ containing $\cos\theta$ and not $\sin\theta$. A further obvious property is that, should such a machine be turned upside down, with the fins below, it would become unstable and tend to right itself.

Case II. **Both fins at the level of the centre of gravity.**

86. If we put $y=0$ and $M_1=P=0$, the condition of stability $\mathfrak{C}>0$ reduces when $\theta=0$ to $x\cos a - y\sin a>0$, that is, the centre of pressure of the two fins must be in

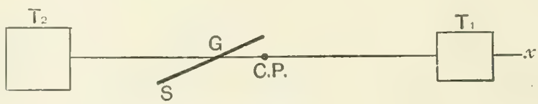

Fig. 33.

front of the centre of gravity (Fig. 33). To make $\mathfrak{D}$ positive in this case, it will be necessary to have

$$\frac{Wx}{Kg} < M_2 \quad \ldots\ldots \quad (154)$$

The stability, as dependent on $\mathfrak{C}$, will in this case be limited by the condition

$$\tan\theta < 2\tan a.$$

Now it may be true that this condition would usually be satisfied in uniform flight, since even if a plane were gliding down uniformly under gravity, θ would only be equal to a, and any value of θ greater than a could only be consistent with *steady motion* if the machine were retarded by reversing the engines or by a sufficiently great head resistance. At the same time, we have to consider the possibility of a machine diving down at a steep angle either in order to recover lost speed or as the result of a sudden gust of wind, and we have further to remember that the conditions of *lateral stability* are

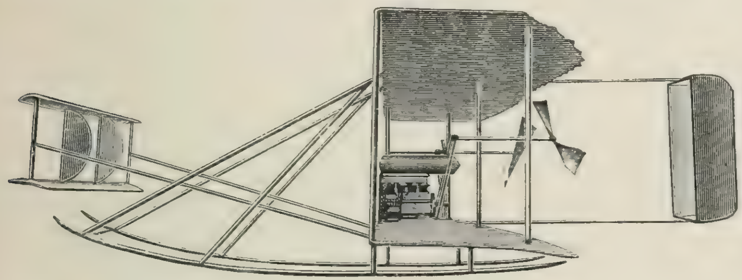

FIG. VI.—DIAGRAMMATIC VIEW OF A WRIGHT BIPLANE USED AT PAU.

An aeroplane having vertical auxiliary fins situated fore and aft of its centre of gravity. From the figure it appears probable that the centre of pressure of the fins in this case was rather behind and not much above the centre of gravity, and that the arrangement failed to secure lateral stability for this reason.

independent of those for *longitudinal equilibrium.* The conclusion is that, if for any reason such a machine should dive downwards at an angle greater than that consistent with the relation $\tan \theta < 2 \tan a$, it would lose its lateral stability, and would probably turn round sideways. This might result in an accident, either through the machine becoming uncontrollable or through breakage caused by undue strains. The erroneous impression might at least be produced that the machine had suddenly been struck by a side gust of wind, causing it to overturn sideways. A knowledge of the risk and its cause might enable an aviator to save the situation.

87. Using the approximate forms of § 80 (which, be it remembered, assume $F=0$), we find

$$\frac{1}{K^2U^2g^2}\left\{\mathfrak{C} - \frac{\mathfrak{A}\mathfrak{D}}{\mathfrak{B}}\right\} = TWB + WT(Tx^2 + M_2) \quad . \quad . \quad (155)$$

and the stability condition $\mathfrak{H}>0$ reduces to

$$M_2 > \frac{Wx}{Kg}\left[1 + \frac{WB\cos\theta\,(2\tan a - \tan\theta)}{KU^2[TB + W(Tx^2 + M_2)]}\right] \quad . \quad (156)$$

This condition can be put in various forms or replaced by a simpler condition, which is sufficient, although not necessary for stability, by omitting some of the terms in the denominator of the last fraction. For example, omitting $Tx^2 + M_2$, we see that stability will be assured by taking

$$M_2 > \frac{Wx}{Kg}\left[1 + \frac{W\cos\theta}{KU^2T}(2\mu - \tan\theta)\right] . \quad . \quad (156a)$$

which from the equations of equilibrium gives

$$M_2 > \frac{Wx}{Kg}\left[1 + \frac{S'\mu}{T}(2\mu - \tan\theta)\right] \quad . \quad . \quad (156b)$$

Or, again, we may write

$$M_2 = T\xi^2, \quad B = Wk_2^2,$$

where ξ is the radius of gyration of the areas of the fins about their centroid, and k_2 is the radius of gyration of the mass of the machine about the axis of y. In this case, the condition reduces to

$$T\xi^2 > \frac{Wx}{Kg}\left[1 + \frac{S'\mu}{T}(2\mu - \tan\theta)\frac{k_2^2}{k_2^2 + x^2 + \xi^2}\right] \quad . \quad (156c)$$

In any case, *stability is secured by making* M_2 *large, and* x *small, that is, by making the distance between the fore and aft fins sufficiently large, and placing the centre of pressure not too much in advance of the centre of gravity.* At the same time, x *must not be too small, otherwise* $\mathfrak{C}$ will *become too small.*

Further, *if* $\tan\theta$ *is negative, or the aeroplane is rising, the condition of stability* $\mathfrak{H} > 0$ *becomes more difficult to*

satisfy as the angle of rising increases, while, as we have already shown, stability ceases when the angle of descent becomes greater than that given by $\tan\theta = 2\tan\alpha$.

The existence of two limits is a direct consequence of the fact that $\mathfrak{C}$ occurs with a negative sign in $\mathfrak{H}$. The present system is one in which a numerical calculation of the actual oscillations obtained by solving the biquadratic for a machine of given dimensions would be of considerable interest. This suggests a subject for future investigation.

Case III. One fin above and one behind of the centre of gravity.

88. The following investigation of this case is due to Mr. Harper. We suppose the fins to be of areas T_1 at the point $(0, y)$ and T_2 at $(x, 0)$ (Fig. 34), and we clearly have

$$\frac{Z_w}{KU} = T_1 + T_2 \qquad \frac{Z_p}{KU} = T_1 y \qquad \frac{Z_q}{KU} = -T_2 x$$
$$\frac{L_w}{KU} = T_1 y \qquad \frac{L_p}{KU} = I' + T_1 y^2 \qquad \frac{L_q}{KU} = -2I'\mu$$
$$\frac{M_w}{KU} = -T_2 x \qquad \frac{M_p}{KU} = -I'\mu \qquad \frac{M_q}{KU} = 2I'\mu^2 + T_2 x^2 \tag{157}$$

The approximate coefficients in the biquadratic where μ is small will be given by

$$\mathfrak{A} = W\{AB - F^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{158a}$$

$$\frac{\mathfrak{B}}{KUg} = \underline{WBI'} + W(By^2T_1 + Ax^2T_2) + (T_1 + T_2)\{AB - F^2\} \tag{158b}$$

$$\frac{\mathfrak{C}}{K^2U^2g^2} = BI'(T_1 + T_2) + \underline{WI'x^2T_2}$$
$$+ T_1T_2(Ax^2 - 2Fxy + By^2 + W^2x^2y^2) - \frac{W}{Kg}(AxT_2 - FyT_1) \tag{158c}$$

$$\frac{\mathfrak{D}}{K^3U^3g^3} = I'T_1T_2x^2 - \frac{W}{Kg}(I' + T_1y^2)xT_2$$
$$+ \frac{W}{K^2U^2g}\cos\theta\,(FxT_2 - \underline{ByT_1}) + \frac{W}{K^2U^2g}\sin\theta\,(FyT_1 - AxT_2) \tag{158d}$$

$$\frac{\mathfrak{E}}{K^4U^4g^4} = \frac{WI'\cos\theta}{K^2U^2g}(2\mu - \tan\theta)(xT_2 - yT_1\mu)$$
$$- \frac{W}{K^2U^2g}(\underline{x\cos\theta} + y\sin\theta)xyT\,T_2 \quad . \quad . \quad . \quad . \quad . \tag{158e}$$

We suppose $F = 0$. Also if the fins are *above* and *behind* the centre of gravity, x and y will both be negative. In these circumstances, all the coefficients except $\mathfrak{C}$ are essentially positive. Putting $\theta = 0$, we see that the condition $\mathfrak{C}$ positive is secured by making

$$xyT_1 > 2I\mu \text{ or } 2I \sin a \cos a \quad . \quad . \quad . \quad (159)$$

which is equivalent to the condition

$$xL_w + L_q \text{ positive} \quad . \quad . \quad . \quad . \quad (159a)$$

It will thus be seen that *for stability the distance of the tail fin behind the centre of gravity must not be less than a certain inferior limit.* It will be noticed, moreover,

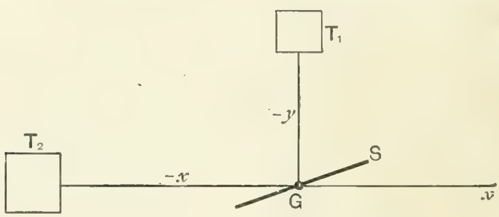

FIG. 34.

that the coefficient of $\sin \theta$ is positive in $\mathfrak{C}$, consequently *stability from this cause is increased when flying downwards, though a limit for stability occurs in flying upwards at too steep an angle.*

89. Coming next to the approximate condition $\mathfrak{C}\mathfrak{D} - \mathfrak{C}\mathfrak{B} > 0$, we notice that for $\theta = 0$ the only positive part of $\mathfrak{C}$ is the term underlined, and if the fin area is small, the term underlined in $\mathfrak{B}$ need alone be retained for a first approximation. The product of these terms cancels out the product of the terms underlined in $\mathfrak{C}\mathfrak{D}$, and hence in $\mathfrak{C}\mathfrak{D} - \mathfrak{C}\mathfrak{B}$, all the terms which are not small in comparison with other terms are positive. The approximate discriminantal condition of stability is thus satisfied subject to the assumptions that have been made.

Mr. Harper further points out that the stability condi-

tion $xL_w + L_q$ positive represents the condition that *if the machine receives a small angular velocity of rotation about a vertical axis passing through the tail plane* $(x, 0)$, *the moment of the air resistances that is set up tends to cause the machine to heel over about the axis of*

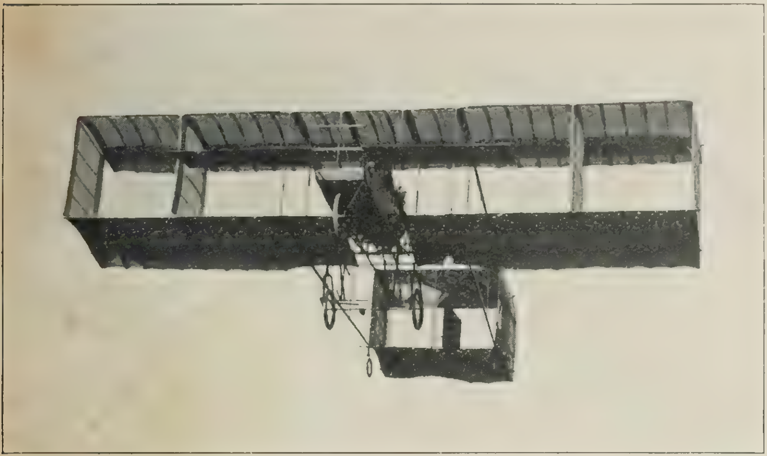

Photo.] [*The Sport and General Illustrations Co.*

FIG. VII.—ROUGIER FLYING AT BLACKPOOL.

The Voisin type with its box-like arrangement of vertical fins is equivalent, so far as lateral stability is concerned, to a system with one fin above and one fin behind the centre of gravity, the conditions of stability of which are discussed in § 88.

x *towards the outside of the curve that it would describe in virtue of the rotation.*

Effect of two or more transverse planes.

90. We have already referred to the case where, instead of a single main plane, there are two planes the moments of inertia of which are I_1, I_2, and the angles of attack of which are α_1, α_2. The differences in this case (and the argument applies to any number of planes) may be conveniently exhibited by the following artifice :—

Construct the vector sum of two vectors $(I_1, 2\alpha_1)$ and

$(I_2, 2a_2)$ and let it be $(I, 2a)$. Then in the formulæ for the resistance derivatives we have

$$\left.\begin{array}{l} I_1 \cos^2 a_1 + I_2 \cos^2 a_2 = \frac{1}{2}(I_1 + I_2 - I) + I \cos^2 a \\ I_1 \sin^2 a_1 + I_2 \sin^2 a_2 = \frac{1}{2}(I_1 + I_2 - I) + I \sin^2 a \\ I_1 \sin a_1 \cos a_1 + I_2 \sin a_2 \cos a_2 = I \sin a \cos a. \end{array}\right\} \quad (160)$$

It follows that *the values of* L_q *and* M_p *are the same as for the single plane* (I, a), *while the values of* L_p *and* M_q *are increased by the same amount as they would be if the moments of inertia* M_1, M_2 *of the fins were each increased by* $\frac{1}{2}(I_1 + I_2 + I)$.

The effect will in general be to increase the stability particularly in such arrangements as those of Cases I and II above. Of course, if $a_1 - a_2$ is small, $I_1 + I_2 - I$ will be small, and generally of the second order in $a_1 - a_2$. The increase of stability will depend mainly on the equivalent increase in M_2, in view of the fact that the value of M_1 has little effect on the stability.

Effects of structural and head resistances and twin screws.

91. As in § 57, the most general effect of air resistances acting on the framework, tangential forces such as friction on the aeroplanes, and other resistances, will be to introduce additional terms into the resistance derivatives which we shall denote by accented letters $(Z', L', M')_{w, p, q}$. In an ordinary flying machine some of these terms will probably be small or unimportant, and it will probably be only possible to make rough and ready estimates of their value. In this connection, experiments on models afford the most desirable tests.

When the machine is turning about the axis of y with angular velocity q, the parts of the framework which are moving with the greatest velocity encounter the greatest resistance, and this circumstance, as well as the resistances on the fuselage and framework supporting the tail, give

rise to the positive coefficient M'_q. This coefficient divided by KU must be added to the determinant (143) of § 79 in the same term as the moment of inertia M_2 of the fins. The effect of the moment of the resistances is thus equivalent to an increase in M_2, and from what has already been shown this increase generally increases the stability.

A similar effect occurs in a machine furnished with twin screws (like the Wright machine), arising from the fact that in turning round, the outer propeller moves through the air faster than the inner one. If the one that is moving quickest exerts the least thrust, as assumed in § 57 (*b*), the effect will be to produce a further couple tending against the rotation q and further increasing the value of M'_q.

From elementary considerations, it should be pretty obvious (without further discussion) that if the speed U of the machine is not largely in excess of that required for maximum efficiency, so that the head resistance is not large compared with the drift (see § 57 *a*), the value of M'_q will not be large with $2KUI\sin^2 \alpha$ or $2KUI'\mu^2$. In a reasonably designed machine, L'_p will be of the same order of magnitude, and will thus be small compared with $KUI\cos^2 \alpha$ or KUI'. This property is equivalent to the obvious statement that if the machine receives a small rotation p about the axis of x, the moment of the air resistances on the framework is small compared with the moment of the air resistances on the main planes. It follows that the coefficient L'_p has generally a negligible effect on the stability. The same thing is evident if we notice that the presence of this added term in the middle term of the determinant (143) is equivalent to an increase in the moment of inertia M_1, which has but little effect on the stability.

As regards the terms M'_p and L'_q, these will probably either vanish or be very small even in comparison with

L'_p and M'_q in the case of a well-designed flying machine. At the most, they will be comparable with $2KUI\mu^2$ and small compared with the corresponding terms $KUI\mu$ dependent on the main planes. If they are equal to one another their effect will be equivalent to a change in the product of inertia P of the fins. In any case, there would be no difficulty in taking account of them should the necessity arise, but this appears hardly probable.

It will not do rashly to neglect effects of resistance represented by Z'_w, Z'_p, Z'_q, L'_w, M'_w. These include the effects of lateral resistance of the fuselage and framework. For a machine with a low centre of gravity, Z'_p and L'_w will evidently be important; for a machine with a long tail, Z'_q and M'_w will be important. The terms must not be neglected, because they occur in the small terms of the determinantal equation for λ, and they are only associated with terms depending on the fins which are also in general small. If the new coefficients satisfy the reciprocal relations $Z'_p = L'_w$ and $Z'_q = M'_w$, their effects can be represented by means of vertical fins. For instance, Z'_p and L'_w are then equivalent to an increase $Z'_p KU$ in the value of Tx, and, if positive, have thus the same effect as shifting the centre of pressure of the fins forward, and so on. This interpretation may perhaps be more useful than a direct calculation of the effects of these resistances. At the same time, when the determinantal equation is developed and the coefficients are expanded, it is sufficient for a first approximation to retain only the terms of lowest significant order in the resistance derivatives in question, as done in the case of fins in (145).

Probably the reciprocal relations $Z'_p = L'_w$ and $Z'_q = M'_w$ are satisfied in almost every case, but though further discussion thus becomes rather uninteresting, it may be pointed out that if these relations are not satisfied it is still possible to represent Z'_w, L'_w, M'_w by suitable

increases in T, Ty, $-Tx$, and to allow for the difference by writing $T(y+\eta)$ and $T(x+\xi)$ for Ty and Tx in the second and third columns of the first row of the determinant (143). The only *important* additional term introduced into the coefficients of the biquadratic is a term $-Tx\xi I'$ in the value of $\mathfrak{D}'K^3U^3g^3$, other changes being negligible for a first approximation.

There is one other way in which the propeller may affect stability apart from gyrostatic action. The *torque* produced by the propeller affects the lateral *equilibrium*, and this, of course, indirectly influences stability, but it is further possible that when the aeroplane possesses a small angular velocity p about the axis of x, the propeller torque is slightly altered, being increased when the aeroplane rotates in the same direction as the propeller, decreased when it rotates in the opposite direction. This effect would be represented by an additional positive term added to L_p. We have seen, however, that such an addition would in general have little effect on the stability.

Effects of friction and curvature of main surfaces.

92. Making use of the property of *narrow planes* assumed in § 76, we may easily deduce the corrections analogous to those of § 69 applicable to lateral stability. In the first place, we observe that if the angle of attack a (defined by $R=KSU^2 \sin a$) is not the same as the angle a' (defined by the property that R makes an angle $90^\circ - a'$ with the axis of x), then supposing a' to remain constant, we write $I'=I \cos a \cos \alpha'$, $\mu = \tan a$, $\mu' = \tan a'$, and the resistance derivatives become

$$L_p = KUI',\ L_q = -2KUI'\mu,\ M_p = -KUI'\mu',\ M_q = 2KUI'\mu\mu' \tag{161}$$

The effects of concavity of the main surface may be investigated by applying the considerations of § 71 to a strip dS of the main plane, the legitimacy of this step being implied in the definition of *narrow planes*. If the machine receives an angular velocity p, about the axis of x, and we consider an element dS, distant z from that axis, the change in the angle of attack is $d\boldsymbol{a} = -zp/U$, the centre of pressure of dS shifts forward through a distance $a\phi'(a)da$, and c being the radius of curvature, the direction of R on the element changes by an amount $d\boldsymbol{a}' = -a\phi'(\boldsymbol{a})d\boldsymbol{a}/c$, the resolved part of R along the axis of x changes by an amount $R \cos \boldsymbol{a}' \, d\boldsymbol{a}'$, and its moment about Oy by $-Rz \cos \boldsymbol{a}' \, d\boldsymbol{a}'$. The final result is to add to M_p a term

$$= \int KUa/c\phi'(a) \sin a \cos a' \, z^2 dS \quad . \quad . \quad . \quad (162)$$

If the plane be rectangular and of constant curvature throughout its span, the corrected expression for M_p becomes

$$M_p = - KUT\{\mu' - \mu\phi'(a)a/c\} \quad . \quad . \quad . \quad . \quad (163)$$

There is a small corresponding change in L_p, which now becomes

$$L_p = KUT\{1 + \mu\mu'\phi'(a)a/c\} \quad . \quad . \quad . \quad . \quad (164)$$

but it will be seen that with μ and μ' small this correction is negligible, the correction in M_p being alone important. These modified forms are only based on the assumption that $a\phi'(\boldsymbol{a})/c$ is constant throughout the span of the planes, so that this factor can be removed outside the sign of integration. This would still be true if the planes were not rectangular, provided that all sections parallel to the plane of x, y formed circular arcs of the same *angle*. It must be remembered that if the centre of pressure moves forward as the angle of attack decreases, $\phi'(\boldsymbol{a})$ is negative.

The correction does not affect the value of $\mathfrak{B}$ when $F = 0$, and under the same condition the effect on $\mathfrak{C}$

is probably not very important. The most important effects are in $\mathfrak{D}$ and the part of $\mathfrak{E}$ involving $\sin\theta$. The portion of $\mathfrak{E}$ involving $\cos\theta$ will be unaffected.

Corrections for aspect ratio, for effects at extremities of main planes and for wash.

93. In order to simplify the algebra, we have in general assumed that the resistance coefficient K is the same for all the planes under consideration. As a matter of fact, however, K is a function of the aspect ratio, as well as of the camber, and, therefore, in dealing with different surfaces S_1, S_2, we ought to have retained K_1S_1, K_2S_2, where we have written KS_1 and KS_2. It will be seen, however, that the algebra is quite sufficiently complicated as it stands, and the same results may be arrived at by waiting till the final formulæ have been obtained, and then *multiplying each area by a coefficient depending on the aspect ratio.* This is equivalent to reducing all the areas to the same aspect ratio, or, more strictly, to the same resistance coefficient K. The results will hold good provided that the "reduced areas" so obtained be substituted for the actual areas of the planes in the formulæ.

In connection with lateral stability a correction is required owing to the fact that the conditions assumed for *narrow planes at small angles* do not hold good at the extremities of the main planes where the stream lines flow over the tips instead of being, at any rate, approximately two-dimensional. This leads to a correction in the value of I or I' dependent on the forms of the planes at their extremities, and, therefore, to the substitution of a corrected constant for the radius of gyration implied in I and I'. According to Lanchester, *two* such constants are required, which he calls the "aerodynamic and aerodromic radii." If this is the case, one will occur in L_p and

M_p, the other in L_q and M_q. If we denote by I'_1 and I'_2 the corresponding values of I' corrected in accordance with this hypothesis, and if we apply the results to the approximate values given by (145) § 79, we find that I'_1 (depending on the *aerodynamic* radius) replaces I' everywhere, except in the part of $\mathfrak{C}$ involving $\cos\theta$, where I_2 occurs. In $\mathfrak{C}$ we must replace $I'\ (2\mu\cos\theta - \sin\theta)$ by $I'_2\ 2\mu\cos\theta - I'_1\sin\theta$, and assuming θ to be small, the effect of the difference between I'_2 and I'_1 is equivalent to an increase in the angle which the flight path makes with the horizon from θ to θ', where

$$\theta' = \theta + \frac{2\mu(I'_1 - I'_2)}{I'_1} \quad . \quad . \quad . \quad . \quad (165)$$

If Lanchester's aerodromic and aerodynamic radii are equal, this effect vanishes, and the only change required is in the assumed value of I'.

The correction for "wash" when two fins are placed one behind the other at the same level should give little difficulty, if it is needed; but the effect could be avoided by placing the fins at slightly different levels.

CHAPTER VIII.

LATERAL STABILITY—BENT UP PLANES.

Comparison of bent up planes with vertical fins.

94. We now consider such cases as that of a supporting surface consisting of a pair of wings bent up at a dihedral angle, or bent up at their tips. In all such cases the general expressions (136) of § 76 must be adopted as the starting point, and the first point which presents itself for inquiry is how far the effects of bending up are equivalent, in their effect on lateral stability, to the addition of vertical fins.

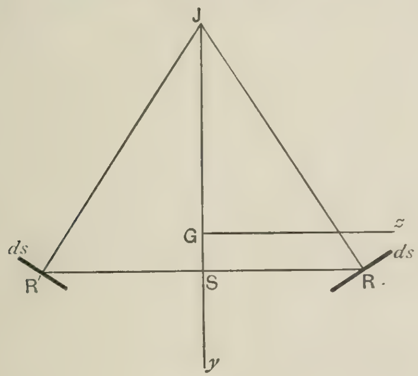

Fig. 35

Consider a pair of elements dS of such planes situated symmetrically at R, R' on opposite sides of the plane of

(x, y), and let the normals to these elements meet the plane of (x, y) in a point J (Fig. 35). Then in every circumstance the resultant pressures on the two elements will pass through J, and will be equivalent to a single force acting through J. The only part of this force which will affect the lateral stability will be its component perpendicular to the plane of (x, y), and thus it might appear that the elements were equivalent to a single vertical fin of suitable area placed at the point J. This, however, is

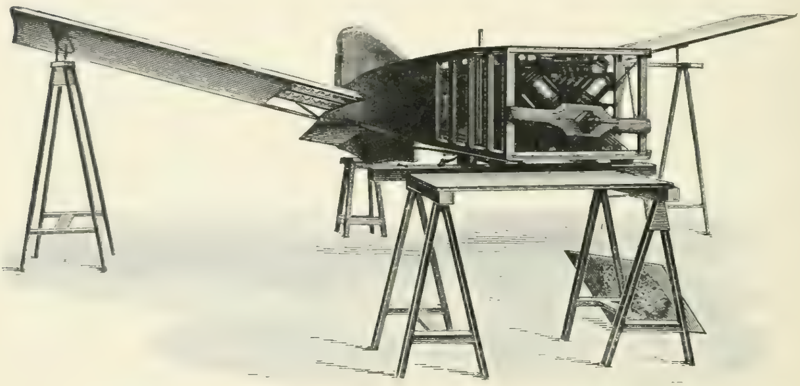

FIG. VIII.—A MONOPLANE IN CONSTRUCTION BY BLÉRIOT IN 1907.

An aeroplane, of the "Antoinette" type, in the course of construction, showing the bent up wings and vertical tail plane in the rear as also the arrangement of the engines, the propeller in front being detached. The photograph was taken by the author in July, 1907, on a visit to Blériot's factory outside the Porte Maillot, Paris, with the late Captain Ferber, before aeroplanes had flown or Blériot had become well known.

not strictly the case as the velocities of R and R' are in general different from that of J.

If ξ, η, 0 be the co-ordinates of J, the equation of the normal gives

$$\frac{x-\xi}{l} = \frac{y-\eta}{m} = \frac{z-0}{n} \quad . \quad . \quad . \quad . \quad (166)$$

whence $lz - nx = -n\xi$, $ny - mz = n\eta$; and therefore $2lz - nx = -2n\xi + nx = -n\xi + lz$.

Substituting in the formulæ of § 76 we get

$$\frac{Z_w}{KU} = \int n^2 dS, \quad \frac{Z_p}{KU} = \int n^2 \eta dS, \quad \frac{Z_q}{KU} = -\int n^2(2\xi - x)dS$$
$$\frac{L_w}{KU} = \int n^2 \eta dS, \quad \frac{L_p}{KU} = \int n^2 \eta^2 dS, \quad \frac{L_q}{KU} = -\int n^2 \eta(2\xi - x)dS$$
$$\frac{M_w}{KU} = -\int n^2 \xi dS, \quad \frac{M_p}{KU} = -\int n^2 \xi \eta dS, \quad \frac{M_q}{KU} = +\int n^2 \xi(2\xi - x)dS \tag{167}$$

The effect of the element dS is thus equivalent to that of an element n^2dS in the plane $z = 0$ at the point $(\xi, \eta, 0)$ in the terms of the first two columns, but in the third column this is no longer the case, the factor $2\xi - x$ occurring in place of ξ. The equivalence will be exact in the case when $\xi = x$. This requires that $l = 0$, *i.e.* that the element should be placed parallel to the axis of x, and therefore in a neutral direction to the line of flight, in which case it will exert no lifting force in steady flight. The simplest application of this property consists in replacing a vertical fin placed above the main planes by two small bent up planes situated at the extremities of these planes, sloping up at an angle β to the horizon, and parallel to the line of flight. In this case, we shall have $l = 0$, $m = \cos\beta$, $n = \sin\beta$, and if σ is the area of either plane, the two will be equivalent to a single fin of area $2\sigma \sin^2\beta$ placed at a height $z \cot\beta$ above them. Such a pair of terminal planes may be called "stabilizers," the term being used whether they are neutral to the line of flight or not.

A pair of tandem planes each fitted with neutral stabilizers will thus be equivalent to a system with two raised vertical fins, thus affording a convenient substitute for the system of § 84, which possesses such wide limits of stability. The substitution of stabilizers for fins, of course, presents obvious advantages not directly connected with the present investigation.

95. In the general case, it is convenient to put

$$l = \sin a \qquad m = \cos a \cos\beta \qquad n = \cos a \sin\beta \quad . \quad . \tag{168}$$

In this case, a is the *angle of attack* or angle made by the element with the direction of steady flight, β the angle through which the element is bent up about an axis parallel to the axis of x. We shall also, in accordance with our usual convention, write

$$dS' = dS \cos^2 a, \ \mu = \tan a$$

and we get

$$\begin{aligned} ny - mz &= \cos a\,(y \sin\beta - z\cos\beta) \\ lz - nx &= \cos a\,(\mu z - x\sin\beta) \\ 2lz - nx &= \cos a\,(2\mu z - x\sin\beta), \end{aligned}$$

and writing $\zeta = z\cos\beta - y\sin\beta$, the expressions for the derivatives become

$$\begin{aligned}
&\frac{Z_w}{KU} = \int \sin^2\beta\, dS', && \frac{Z_p}{KU} = -\int \sin\beta\,\zeta dS', \quad \frac{Z_q}{KU} = \int \sin\beta(2\mu z - x\sin\beta)dS' \\
&\frac{L_w}{KU} = -\int \sin\beta\,\zeta dS', && \frac{L_p}{KU} = \int \zeta^2 dS', \quad \frac{L_q}{KU} = -\int \zeta(2\mu z - x\sin\beta)dS' \\
&\frac{M_w}{KU} = \int \sin\beta(\mu z - x\sin\beta)dS', && \frac{M_p}{KU} = -\int \zeta(\mu z - x\sin\beta)dS', \\
& && \frac{M_q}{KU} = \int (2\mu z - x\sin\beta)\,(\mu z - x\sin\beta)dS'
\end{aligned} \tag{169}$$

These expressions and the subsequent analysis become considerably simplified when $x \sin\beta$ is negligible compared with μz.

This occurs (1) in the case of a single lifting system with bent up wings in which $x = 0$, (2) in the case of a tandem system of bent up lifting planes, when the angle of bending up β is small compared with the angle of attack a, and the span of the planes is not small compared with the distance between them.

Single lifting plane with terminal stabilizers.

96. The following solution, due to Mr. Harper, applies to the case of a single straight lifting surface S, furnished at the ends with a pair of small stabilizers of total area T

(each being of area $\frac{1}{2}T$), placed with their centres in the plane $x=0$, at the points $R\,(0, y, z)$ and $R'\,(0, y, -z)$, their angles of attack being α' and the stabilizers being bent up at an angle β. In these circumstances, the traces of the stabilizers in the plane $x=0$ will make angles $\pm\beta$ with the axis of z, and if the direction of the plane at R

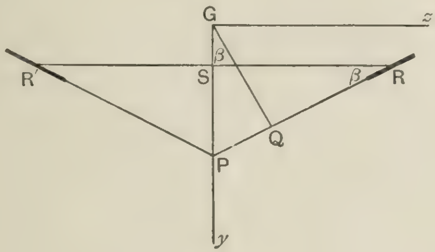

FIG. 36.

be produced to meet Gy in P, and GQ be drawn perpendicular on RP (Fig. 36), we shall have

$$RQ = z\cos\beta - y\sin\beta = \zeta,$$

with the notation of the last article. *The angle β must obviously not be too small if the stabilizers are to be efficient.*

We shall, as usual, employ the notation $T\cos^2\alpha' = T'$, $\mu' = \tan\alpha'$, which greatly simplifies the analysis, and further to shorten matters, we shall assume, in the first instance, that K is the same for S and T, *it being distinctly understood* that differences in this respect are to be corrected for in any future formulæ by multiplying T or T' in the final result by the ratio K'/K of the two coefficients, and that this correction may, in certain circumstances, be considerable. It only remains to write μ'/T' for $\mu dS'$ and $x=0$ in the results of the last article, to remove the sign of integration, and to add the results to those of § 76 due to the lifting plane.

Thus the equation in λ becomes

$$\begin{vmatrix} \dfrac{W\lambda}{KUg} + T' \sin^2\beta, & \dfrac{W\cos\theta}{K^2U^2g}\left(\dfrac{KUg}{\lambda}\right) - T'\zeta\sin\beta, & -\dfrac{W}{Kg} - \dfrac{W\sin\theta}{K^2U^2g}\left(\dfrac{KUg}{\lambda}\right) + 2T'\mu' z\sin\beta \\ -T'\zeta\sin\beta, & \dfrac{A\lambda}{KUg} + I' + T'\zeta^2, & -\dfrac{F\lambda}{KUg} - 2I'\mu - 2T'\mu' z\zeta \\ T'\mu' z\sin\beta, & -\dfrac{F\lambda}{KUg} - I'\mu - T'\mu' z\zeta, & \dfrac{B\lambda}{KUg} + 2I'\mu^2 + 2T'\mu'^2 z^2 \end{vmatrix} = 0 \quad . \quad . \quad (170)$$

and the coefficients are given by

$$\mathfrak{A} = W(AB - F^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171a)$$

$$\frac{\mathfrak{B}}{KUg} = T'\sin^2\beta(AB - F^2) + W\{2A(I'\mu^2 + T'\mu'^2z^2) - 3F(I'\mu + T'\mu' z\zeta) + B(I' + T'\zeta^2)\} \quad (171b)$$

$$\frac{\mathfrak{C}}{K^2U^2g^2} = T'I'\sin^2\beta(2A\mu^2 - 3F\mu + B) + 2WI'T'(\mu\zeta - \mu' z)^2 + \frac{W}{Kg}T'\sin\beta(A\mu' z - F\zeta) \quad . \quad . \quad . \quad . \quad . \quad (171c)$$

$$\frac{\mathfrak{D}}{K^3U^3g^3} = -\frac{W}{Kg}T'I'\sin\beta(\mu\zeta - \mu' z) + \frac{W}{K^2U^2g}T'\sin\beta[(A\mu' z - F\zeta)\sin\theta - (F\mu' z - B\zeta)\cos\theta] \quad (171d)$$

$$\frac{\mathfrak{E}}{K^4U^4g^4} = \frac{W}{K^2U^2g}T'I'\sin\beta(\mu\zeta - \mu' z)(2\mu\cos\theta - \sin\theta) \quad . \quad . \quad (171e)$$

The condition of stability $\mathfrak{E}$ positive in the case of horizontal flight requires that ζ *shall be positive and greater than* $\mu' z/\mu$, that is, in the figure,

$$RS\tan\alpha' < RQ\tan\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

and *stability necessarily ceases when the angle of descent passes the limit given by*

$$\tan\theta = 2\tan\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

The condition RQ positive is satisfied if the normals to R and R' meet above the centre of gravity, not below.

The above condition makes the first term of $\mathfrak{D}$ negative. In the case of $\theta = 0$ and $F = 0$, the condition $\mathfrak{D}$ positive leads to

$$-\mu\zeta + \mu' z + \frac{B}{I'KU^2}\zeta > 0,$$

that is,

$$RS\tan\alpha' > RQ\left(\tan\alpha - \frac{B}{I'KU^2}\right) \quad . \quad . \quad . \quad (174)$$

If

$$B > I'KU^2 \tan a,$$

that is,

$$B > KU^2 I \cos a \sin a \quad . \quad . \quad . \quad . \quad (175)$$

this condition (174) is satisfied for all positive values of a' (RQ being positive) and for $a' = 0$, in which case the planes would be neutral.

Now, since by the condition of equilibrium

$$W = KU^2 S \sin a \cos a + KU^2 T \sin a' \cos a' \cos \beta,$$

it follows that when T is small compared with S this condition may be written *approximately*,

$$\frac{B}{W} > \frac{I}{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

and that this condition is sufficient even if T is not neglected. This requires that *the radius of gyration of the mass of the aeroplane about the axis of* y *should exceed that of the areas of its main planes about the axis of* x.

If this condition is not satisfied, stability cannot be secured with the stabilizers in the neutral position, and the conditions of stability may be written

$$\frac{RS}{RQ}\frac{\tan a'}{\tan a} < 1, \text{ but } > 1 - \frac{\text{sq. of rad. of gyration of mass about } Oy}{\text{sq. of rad. of gyration of } S \text{ about } Ox} \quad (177)$$

If the stabilizers are attached at the same level as the centre of gravity (S and G coinciding), $RQ = RS \cos \beta$, and RQ must therefore $< RS$ and $a' < a$.

It is further to be observed that *by making the product of inertia* F *negative, both* $\mathfrak{C}$ *and* $\mathfrak{D}$ *can be increased, a condition favourable to increased stability.*

97. The further condition necessary for stability $\mathfrak{H} > 0$ is somewhat complicated, and we shall only consider the approximate form when both T and μ are small, and $F = 0$, $\theta = 0$, μ' being also small from condition (172). In this case

$$\frac{1}{KUg}\frac{\mathfrak{C}}{\mathfrak{B}} \text{ approximates to } \frac{T' \sin^2 \beta}{W},$$

while the term $\mathfrak{A}\mathfrak{D}^2$ becomes negligible, and the condition, which may be written $\mathfrak{G}\mathfrak{B} > \mathfrak{G}\mathfrak{D}$ leads to

$$\frac{2\mu W}{T \sin^2 \beta} < \frac{B\zeta}{I(\mu\zeta - \mu'\zeta')} - KU^2$$

which becomes

$$\mu\zeta - \mu'\zeta' < \frac{B\zeta}{I\left(\frac{2\mu W}{T \sin^2 \beta} + KU^2\right)}$$

that is,

$$RQ \tan \alpha - RS \tan \alpha' < \frac{B \,.\, RQ}{IW\left(\frac{2 \sin \alpha \cos \alpha}{T \cos^2 \alpha' \sin^2 \beta} + \frac{\cot \alpha}{S}\right)} \quad . \qquad (178)$$

(assuming the approximate condition of equilibrium).

It will be observed that the system here considered *does not include a tail plane.* The addition of this would further complicate the algebra, and the case may perhaps stand over for future investigation in view of the discussion of the analogous problem of the next article. The possibility of satisfying the conditions of stability with α' negative offers another interesting question, especially in view of published accounts of the Dunne biplane.

Theory of the Antoinette type.

98. The following investigation is due to Mr. E. H. Harper. By the "Antoinette" type is here meant a machine supported by a single-lifting pair of bent up wings, and provided with a vertical tail or rudder fin. It is stated that in such arrangements it has been found necessary to place the centre of gravity G lower than the intersection P of the wings, and we shall assume that this is done, the tail being, however, on a level with the centre of gravity. We are not concerned with the effect of a horizontal rudder, which only affects the longitudinal stability.

Taking Fig. 37 as representing the section of the wings in the plane of (y, z), R any point on the wing on the positive side of the plane of (x, y), GQ a perpendicular on the wing AP produced, let $RP = \rho$, $PQ = \epsilon$, so that

$$RQ = z\cos\beta - y\sin\beta = \rho + \epsilon, \quad RP = z\sec\beta = \rho.$$

Also we suppose the rudder of area T placed at $(x, 0, 0)$, x being negative if the rudder is behind.

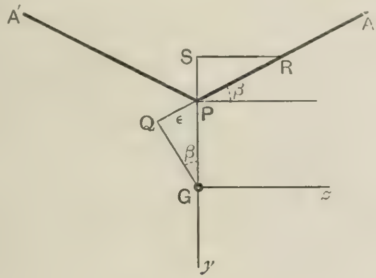

Fig. 37.

With this notation the determinantal equation becomes

$$\begin{vmatrix} \dfrac{W\lambda}{KUg} + \displaystyle\int \sin^2\beta dS' + T, & \dfrac{W\cos\theta}{K^2U^2g}\left(\dfrac{KUg}{\lambda}\right) & -\dfrac{W}{Kg} - \dfrac{W\sin\theta}{K^2U^2g}\left(\dfrac{KUg}{\lambda}\right) \\ & -\displaystyle\int(\rho+\epsilon)\sin\beta dS', & +2\displaystyle\int \mu\rho\sin\beta\cos\beta dS' - xT, \\ -\displaystyle\int(\rho+\epsilon)\sin\beta dS', & \dfrac{A\lambda}{KUg} + \displaystyle\int(\rho+\epsilon)^2 dS', & -\dfrac{F\lambda}{KUg} \\ & & -2\displaystyle\int \mu\rho(\rho+\epsilon)\cos\beta dS', \\ \displaystyle\int \mu\rho\sin\beta\cos\beta dS' - xT, & -\dfrac{F\lambda}{KUg} & \dfrac{B\lambda}{KUg} + x^2T \\ & -\displaystyle\int \mu\rho(\rho+\epsilon)\cos\beta dS', & +2\displaystyle\int \mu^2\rho^2\cos^2\beta dS', \end{vmatrix} = 0 \qquad (179)$$

Assuming the hypotheses involved in the definition of narrow planes, these expressions would hold good even if the angles of attack and of bending up α and β should vary for different parts of the wing, as in the wings of birds, provided ρ and ϵ as well as α and β are treated as varying at different distances from the plane of (x, y) and are properly defined, ρ not being the distance of R from P,

but the length of the tangent at R measured to the plane $z=0$. Except in the case of numerical calculations, however, the investigation would probably be far too complicated to lead to any simple results. We therefore consider only the case where α, β, ϵ are constant. In this case we may write

$$\int(\rho + \epsilon)^2 dS' = S'[(\bar{\rho} + \epsilon)^2 + \kappa^2] \text{ (etc.)},$$

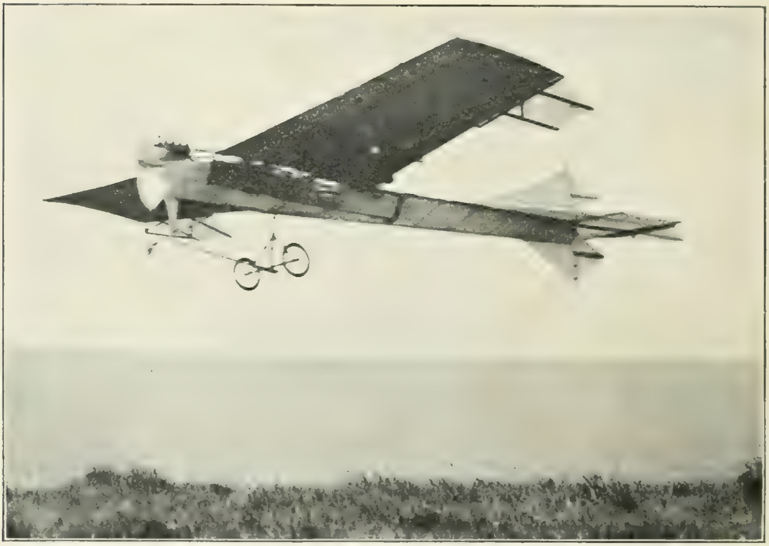

Photo.] [*The Sport and General Illustrations Co.*

FIG. IX.—LATHAM'S TRIAL FLIGHT.

An example of the "Antoinette" type in which *lateral stability* is obtainable by means of (i) bent up wings, (ii) centre of gravity below the dihedral angle formed by the wings (iii) a vertical tail fin placed at not less than a certain limiting distance behind the main planes (§§ 98–101). For longitudinal stability the aeroplane may probably be regarded as a "single lifting system with neutral tail," though it may be desirable to apply corrections for displacement of centre of pressure and camber.

where $\bar{\rho}$ refers to the centroid of the positive wing, and κ is its radius of gyration about that point. It will easily be seen that

$$\int \rho^2 dS' \int dS' - \left(\int \rho dS'\right)^2 = S'^2 \kappa^2$$

and the other minors of the determinant all simplify in

the same way. With this simplification alone, the other terms being left as integrals, the coefficients of the biquadratic become

$$\mathfrak{A} = W(AB - F^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180a)$$

$$\frac{\mathfrak{B}}{KUg} = W\left\{2A\mu^2\int\rho^2\cos^2\beta dS' - \underline{\underline{3F\mu\int\rho(\rho+\epsilon)\cos\beta dS'}} + \underline{B\int(\rho+\epsilon)^2 dS'}\right\}$$
$$+ (AB - F^2)S'\sin^2\beta. \quad + T\{(AB - F^2) + WAx^2\} \quad . \quad (180b)$$

$$\frac{\mathfrak{C}}{K^2U^2g^2} = [(2A\mu^2\cos^2\beta - \underline{\underline{3F\mu\cos\beta}} + B)\sin^2\beta + 2W\epsilon^2\mu^2\cos^2\beta]S'^2\kappa^2$$
$$+ \frac{W\sin\beta}{Kg}\left[A\mu\cos\beta\int\rho dS' - F\int(\rho+\epsilon)dS'\right]$$
$$+ T\Big[\underline{Wx^2\int(\rho+\epsilon)^2 dS'} + A\int(2\rho^2\mu^2\cos^2\beta + 3\mu\rho x\sin\beta\cos\beta + x^2\sin^2\beta)dS'$$
$$- F\int(3\mu\rho\cos\beta + 2x\sin\beta)(\rho+\epsilon)dS' + B\int(\rho+\epsilon)^2 dS' - \frac{W}{Kg}Ax\Big]$$
$$(180c)$$

$$\frac{\mathfrak{D}}{K^3U^3g^3} = -\frac{W}{Kg}\epsilon\mu\sin\beta\cos\beta S'^2\kappa^2$$
$$+ \frac{W\sin\beta}{K^2U^2g}\left\{(A\sin\theta - F\cos\theta)\mu\cos\beta\int\rho dS' + (B\cos\theta - F\sin\theta)\int(\rho+\epsilon)dS'\right\}$$
$$+ T\Big[(2\epsilon^2\mu^2\cos^2\beta - 3x\epsilon\mu\sin\beta\cos\beta + x^2\sin^2\beta)S'^2\kappa^2 - \frac{W}{Kg}x\int(\rho+\epsilon)^2 dS'$$
$$+ \frac{W}{K^2U^2g}x(F\cos\theta - A\sin\theta)\Big] \quad . \quad . \quad . \quad . \quad (180d)$$

$$\frac{\mathfrak{E}}{K^4U^4g^4} = \frac{W\epsilon}{K^2U^2g}(2\mu\cos\theta\cos\beta - \sin\theta)S'^2\kappa^2\mu\sin\beta\cos\beta$$
$$+ \frac{WT}{K^2U^2g}\Big[\underline{x^2\cos\theta\sin\beta\int(\rho+\epsilon)dS'} + 2x\mu\cos\theta\cos\beta\int\rho(\rho+\epsilon)dS'$$
$$- x\sin\theta\int(\rho+\epsilon)^2 dS'\Big] \quad . \quad . \quad . \quad . \quad . \quad (180e)$$

99. Case I.—**Let T = 0** (no rudder tail).

In $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, all the terms are positive except some containing first powers of F. These vanish when $F=0$, and, moreover, the terms underlined twice in F are small compared with other terms in the coefficients, so that the value of F has little effect on $\mathfrak{B}$. On the other hand a *negative value of* F *will certainly increase both* $\mathfrak{C}$ *and* $\mathfrak{D}$, *and will consequently be favourable to stability.* We shall, however, here suppose that $F=0$.

To make $\mathfrak{E}$ positive when $\theta=0$, we must have $\epsilon\sin\beta$ positive; the condition $\mathfrak{D}$ positive then requires that $\sin\beta$ should be positive, hence ϵ must also be positive, that is,

the wings must bend upwards, not downwards, and their intersection must be above the centre of gravity, not below.

The range of inclination of flight path consistent with stability will be limited by the condition (from $\mathfrak{E} > 0$)

$$\tan\theta < 2\tan a\cos\beta.$$

The condition $\mathfrak{D}$ positive becomes with $F=0$

$$\frac{B\cos\theta + A\mu\cos\beta\sin\theta}{KU^2}\bar{\rho} + \epsilon\left[\frac{B\cos\theta}{KU^2} - S\kappa^2\sin a\cos a\cos\beta\right] > 0 \quad (181)$$

If

$$\frac{B\cos\theta}{KU^2} > S\kappa^2\sin a\cos a\cos\beta \quad . \quad . \quad . \quad (182)$$

the coefficient of ϵ will be positive and the condition $\mathfrak{D} > 0$ will be satisfied for all values of ϵ.

This condition can be written

$$B\cos\theta > KU^2S\sin a\cos a\cos\beta\kappa^2$$

that is from the conditions of equilibrium

$$B > W\kappa^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (182a)$$

or radius of gyration of mass about $0y$

$>$ radius of gyration of each plane about its centroid, in every case, or for rectangular planes of span about $2b$

$$B > Wb^2/12 \quad . \quad . \quad . \quad . \quad . \quad . \quad (182b)$$

If, however, this condition is not satisfied, there is a superior limit to the value of ϵ beyond which the condition $\mathfrak{D} > 0$ fails.

The condition $\mathfrak{H} > 0$, for which the approximation $\mathfrak{C}\mathfrak{D} - \mathfrak{E}\mathfrak{B} > 0$ is probably sufficient in most cases, imposes a further limitation on the value of ϵ. It is obvious that $\mathfrak{C}\mathfrak{D} - \mathfrak{E}\mathfrak{B}$ is positive when $\mathfrak{E} = 0$, which happens if $\epsilon = 0$ and is negative if $\mathfrak{D} = 0$. The conditions may be satisfiable for all values of ϵ by making $B - W\kappa^2$ sufficiently large, otherwise the value of ϵ cannot exceed a certain limit ϵ_2 less than the limit ϵ_1 determined by the condition $\mathfrak{D} > 0$.

The general conclusion is that while the condition of

stability $\mathfrak{C} > 0$ requires the centre of gravity to be below the dihedral angle, too low a position may lead to the failure of the condition $\mathfrak{H} > 0$. The effect of this failure would be that the modulus of decay of the oscillations would become negative, and the oscillations would therefore increase in amplitude. This conclusion is entirely in accordance with the results of experience as stated in aeronautical books and journals, although, owing to the somewhat ill-defined notions that commonly prevail regarding the meaning of stability, the production of dangerous oscillations has up till now not been always attributed to its probable cause, namely, inherent lateral instability.

100. Case II.—**Let** $\epsilon = 0$, stability being due to the rudder T. Let x be negative (rudder behind centre of gravity, β positive.

In this case all the terms in $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, with the exception of a few containing F, are positive. To make the part containing T in $\mathfrak{C}$ positive in the case of $\theta = 0$, it is necessary to make

$$(-x) > \frac{2 \tan a \int \rho(\rho + \epsilon) dS'}{\tan \beta \int (\rho + \epsilon) dS'} \quad . \quad . \quad . \quad . \quad (183)$$

so that, assuming as we do that $\epsilon = 0$, and supposing the wings rectangular and of span $2b$, we must have

$$(-x) > \frac{4b}{3} \frac{\tan a}{\tan \beta} \quad . \quad . \quad . \quad . \quad (183a)$$

that is, the rudder, if placed behind the centre of gravity, must be at a distance behind greater than $\frac{4}{3}\, b \tan a \cot \beta$ in order to ensure stability.

With this arrangement there is usually not much difficulty in securing the additional condition of stability $\mathfrak{H} > 0$. For in (180) when T and β are small $\mathfrak{B}/KUg$ reduces to the singly-underlined part $WB \int (\rho + \epsilon)^2 dS'$, and the positive part of $\mathfrak{C}$, also underlined, when multiplied

by this, cancels out with the product of the two singly-underlined terms in $\mathfrak{E}\mathfrak{D}$. The rest of $\mathfrak{E}\mathfrak{D} - \mathfrak{E}\mathfrak{B}$ is essentially positive, at least with this approximate value of $\mathfrak{B}$.

101. It will thus be observed that raising the wings above the centre of gravity is favourable to the condition $\mathfrak{E}$ positive, but the possibilities in this direction are limited by the conditions $\mathfrak{D}$ and $\mathfrak{H}$ positive. On the other hand the addition of a tail plane is favourable to the latter conditions, but there is an inferior limit to the length imposed by the former. A combination of the two may evidently be advantageous. Another arrangement of considerable interest is that in which two rudder planes are placed one in front of and one behind the main planes so as to make $\Sigma Tx = 0$, and in which, moreover, $\epsilon = 0$. In this case the conditions $\mathfrak{D} > 0$ and $\mathfrak{E} > 0$ will certainly be satisfied.

Range of stability.—So far as ϵ is concerned, $\mathfrak{E}$ diminishes when the aeroplane is descending, while so far as T is concerned, $\mathfrak{E}$ diminishes when ascending. By a suitable combination of raised planes and rudder the two effects may be made to counteract each other.

In $\mathfrak{E}$, the part containing $\sin\theta$ is

$$\frac{W}{K^2U^2g}\left[-\epsilon S'^2\kappa^2\mu\sin\beta\cos\beta - Tx\int(\rho+\epsilon)^2 dS'\right]\sin\theta \quad . \quad (184)$$

and this will vanish if

$$T(-x) = \epsilon S\sin a\cos a\sin\beta\cos\beta\,\frac{\kappa^2}{(\bar{\rho}+\epsilon)^2+\kappa^2} \quad . \quad . \quad (185)$$

giving for rectangular planes and ϵ small,

$$T(-x) = \tfrac{1}{4}\,\epsilon S\sin a\cos a\sin\beta\cos\beta \quad . \quad . \quad . \quad (185a)$$

In this case $\mathfrak{E}$ will be independent of θ, except for the factor $\cos\theta$. If the coefficient of $\sin\theta$, instead of being zero, is made small, the range of inclination consistent with stability will still be correspondingly extended.

CHAPTER IX.

GENERAL CONCLUSIONS.

102. It may, I suppose, be now taken as proved that an aeroplane can be constructed which possesses **inherent stability**, both longitudinal and lateral, when propelled horizontally, and also when the inclination of the flight-path to the horizon does not exceed certain determinate limits. Further, these limits can be extended in various different ways, and the limitations regarding the **angle of descent** can be altogether removed. It will be evident, however, that an aeroplane constructed on these lines differs in certain points of detail from those adopted by many of the most successful of present-day aviators; indeed, it does not appear to be denied that some of these are unstable. Thus arises a divergence of opinion regarding the advantage or otherwise of *inherent stability*, and all that can be done now is to state the case for the side which is supported by the present investigation.

In the case of longitudinal stability, there is probably very little need now to advocate the claims of auxiliary surfaces, for the use of these has by this time become almost, if not quite, universally recognised. Practically every modern aeroplane is provided with tail planes, and many are provided with both front and rear "controls," an arrangement which, as is here shown, gives increased longitudinal stability, and in this connection is especially useful in ascending. The original Wright biplane was

provided only with auxiliary planes in front, and this arrangement could only be made longitudinally stable by throwing the weight forward and inclining these rudder planes at a steeper angle to the line of flight than the main supporting planes, an arrangement which would certainly give increased "drift" or resistance in the direction of motion. Since then a stabilizing tail plane has been added in the machines put on the market.

It is in connection with lateral stability, however, that different counsels prevail, some authorities considering it preferable to avoid the use of bent-up planes, to reduce vertical auxiliary surfaces to the minimum necessary for steering, and, in short, to abandon inherent lateral stability altogether. Among the reasons given for this course it is stated that (1) devices such as fins or bent-up planes are very uncertain in their action; (2) they are liable to set up oscillations; (3) they may cause serious rolling when the aeroplane is suddenly struck by a side wind.

Now it will be seen that the problem of lateral stability is one of considerable complexity, especially when bent-up surfaces are concerned. It is easy to see how an attempt to deal with the matter by trial might lead to uncertain results, the significance of which it would be difficult to understand, *without some kind of theory to act as a guide.* In attempting to avoid one condition of instability, we run the risk of introducing another. If a single vertical rudder only is provided, it introduces one kind of instability when in front and another when behind.

The production of oscillations indicates that the aeroplane is *dynamically unstable*; in other words, that the arrangement instead of producing stability has had the reverse effect. We may illustrate this point in connection with the Antoinette type.

It is stated that the bent up wings necessitate a low centre of gravity, but if this is put too low oscillations

are set up. This is entirely in accordance with Mr. Harper's investigations given here, which show that one condition of stability can be satisfied by placing the centre of gravity below the dihedral angle formed by the wings, but that another condition sometimes requires that its depth should be less than a certain limit. By placing the centre of gravity too far below the dihedral angle the machine may again become unstable, and the kind of dynamical instability that is indicated by the conditions is that in which oscillations of increasing amplitude are set up.

103. *The behaviour of an aeroplane when struck by gusts of wind* depends to a very great extent on its inherent dynamical stability.

A sudden gust of wind which quickly subsides will disturb the motion of the aeroplane, and the important requisite is that it shall quickly settle down into its original steady motion. For this to happen it is essential that the state of steady motion in question shall be dynamically stable; if this is not the case the aeroplane will either deviate further from the equilibrium position, or oscillations will be set up which will continually increase.

A permanent change of wind velocity, say V, will cause an alteration in the conditions of equilibrium, and the aeroplane will have to acquire the steady motion consistent with the new conditions. If the relative velocity of the aeroplane when moving steadily is U, the aeroplane will in future have to move with a relative velocity U and therefore a resultant velocity compounded of U and V. This again involves the condition that the new steady motion shall be stable. The subsequent problem is just the same as if the aeroplane had remained in still air, but had from some cause or other been jerked in such a way as to impart an initial velocity to it equal and opposite to V.

Periodic gusts of wind would, of course, give rise to forced oscillations, thus presenting a further problem for the mathematician, as we have only here considered free oscillations. If the period of the gusts happened to coincide with a period of free oscillation, a swaying motion would be set up which might become dangerous. Now the theory of forced oscillations shows that when the free oscillation is a *damped oscillation*, the forced oscillations do not in these cases increase beyond a certain limit. The damping of the oscillations thus becomes important, and this is secured by giving the aeroplane adequate *dynamical stability*.

104. One difficulty still remains, and shows the need of making the motion of aeroplanes the subject of further mathematical investigations than have been possible in the limits of this book. The investigation of stability may, from what has been said above, be fairly well sufficient when we have to deal with the effects of *light gusts of wind* where the change of wind velocity V is small compared with the velocity U of the aeroplane itself.

It is, however, important that investigations should be made of the dynamical effects of a sudden squall of wind of the most violent character that an aeroplane is liable to encounter in the course of its flight. A further feature of such an investigation should be the calculation of the stresses set up in the framework of the aeroplane by the wind pressures. The problem is one of the many unsolved problems awaiting solution at the hands of mathematicians, of which a list is given at the end.

The important condition to be satisfied in such a case is that the motions initially imparted to the aeroplane shall not be of such a character as to render the machine unmanageable. In this respect, I suppose, we may take it for granted that rotations are the most objectionable motions, and that it is essential that these shall be small. If this be assumed, it is probably an easy matter to

reconcile this condition with the conditions of inherent stability.

Thus if an aeroplane is fixed with a vertical rudder plane in the rear, a side gust of wind will cause it to swerve round and face the wind. If it has the vertical rudder in front, it will turn away from the wind. *If, however, it is provided with fins both fore and aft, and their combined centre of pressure is at the centre of gravity, a side wind will cause no rotation about a vertical axis.*

If the fins be raised *above* the centre of gravity, as in the arrangement of § 84, a side wind will cause the aeroplane to heel over about the line of flight, but the rotation thus produced will be damped by the resistance to rotation depending on the term containing $I \cos^2 \boldsymbol{a}$ in our equations, which is large. In any case, the rotation can be minimised by making the height of the fins not too great, and it will be seen that *this is exactly what the conditions of stability require.*

If, as in § 86, the centre of pressure of the fins is placed slightly *in front* of the centre of gravity, *the conditions of stability require that its distance in front shall not be too large, and this again is in accordance with the requirement that the rotation set up by a side wind shall be small.*

The construction of aeroplanes with fore and aft auxiliary surfaces for *longitudinal* balance and stability, moreover, provides the framework on which it only remains to add the necessary vertical fins required for lateral stability. The Curtiss biplane already has a triangular vertical fin in front and a vertical rudder in the rear.

The use of two raised fins placed fore and aft is no new thing, for such an arrangement is shown by Lanchester in connection with models constructed by him in 1894.

The " Langley Aerodrome " consisted of two pairs of bent-

up wings placed tandem, with a rudder behind. Having regard to the partial analogy between the effects of bent-up wings and vertical fins, this type possessed some resemblance in its properties to the arrangement with two raised fins, and was probably laterally stable. This view is confirmed by the records of its flights. At the same time, the stability of such a type deserves more detailed study, and it is with regret that it is now crowded out of our text and placed in our list of unsolved problems.

CHAPTER X.

COMPARISON WITH OTHER THEORIES.

105. We are here only concerned with investigations directly related to the present work, so that the following remarks only refer in many cases to the particular sections of the writings in question which deal with stability.

Bryan-Williams (1903).[1]

The resistance derivatives were here divided by the mass of the aeroplane. The X_u of this book was therefore then denoted by mX_u. There was a term denoted in our present notation by $N_r(X_u + Y_v)$ omitted in the expression for $\mathfrak{C}$, which was incorrect to this extent.

Captain Ferber.[2]

106. Captain Ferber assumes in the first place that the surfaces of an aeroplane can be replaced by three equivalent "fictive surfaces" S, s, σ in three planes mutually at right angles. This is true in certain cases, but not in the sense implied in his footnote, where he assumes that a surface can be replaced by its projections, and that the error will be small. If instead of an inclined surface, we consider its projections on the three co-ordinate planes, then when the wind is blowing parallel to one of

[1] *Proc. Royal Society*, 73, June, 1903.

[2] *Revue d'Artillerie* 67, i, ii, October and November, 1905.

the axes there will be no pressure in a perpendicular direction such as there was on the original surface.

Ferber's theorem is true in the following sense. We assume that an *ideal aeroplane* is a system built up of non-interfering surface elements, the air resistances on which follow rigorously the sine law for all angles of incidence. Then by applying a similar transformation to that used in elasticity or in dealing with moments of inertia, we can show that when the system is moving *without rotation* there are three mutually perpendicular planes, such that the component forces due to air resistance are the same in magnitude and direction as would be produced if the system were replaced by three areas, S_1, S_2, S_3, situated in these planes. Calling these the principal planes and their intersections principal axes, we can show that if certain conditions are satisfied the equivalence can be extended to the moments of the forces by suitably choosing the positions of the areas S_1, S_2, S_3 in their planes. The conditions require that a wind blowing parallel to a principal axis shall produce no couple about that axis, the exceptions thus being surfaces analogous to screw propellers. For an aeroplane with one plane of symmetry these conditions are satisfied.

When the effects of rotation are taken into account this equivalence no longer holds good. This is shown by our proof that two fins give lateral stability that cannot be obtained with one. Captain Ferber thus neglects the most important effects of rotation in his treatment of lateral stability, namely the terms depending on I, the moment of inertia of the main plane areas. As a consequence, probably of this, he gets a cubic instead of a biquadratic for lateral stability, or as he calls it, an equation of the fifth degree with two zero roots. The extra degree is accounted for by his including the azimuth among the variables, and, as has been pointed out, stability in azimuth does not come into the question here any more

than in other forms of locomotion. The presence of the other zero root shows that there is something wrong with the stability. The neglected term I really is of first importance in the problem when we are dealing with aeroplanes, the surfaces of which have considerable span.

The use of Euler's angular co-ordinates is also somewhat unfortunate, for, as we have pointed out, this system is unsuitable for investigating oscillations about a position of equilibrium, Captain Ferber's equations breaking down if we put $\theta = 0$ (see § 14 of this book).

Lanchester.[1]

107. In most points of fundamental importance there is substantial accord between the present conclusions and Lanchester's, and the main differences are in the method of treatment.

Lanchester's condition of longitudinal stability is based essentially on what we here call the hypothesis of "narrow planes flying at small angles," and it agrees with the result here obtained in the "simplest case" of horizontal flight, although it is probably correct to say that the method of obtaining it has an appearance of being wanting in rigour. Lanchester starts with a study of the "phugoid" curves described by an ideal plane which always places itself tangentially to the direction of motion, drift and rotatory inertia, thus, being neglected. He then applies corrections for resistance and moment of inertia, and in the latter connection takes account of the effects of the auxiliary tail on the rotatory motion. In doing so he assumes the motion to be harmonic, and takes the mean effect over each oscillation instead of writing down a differential equation.

The reason why Lanchester's method leads to correct

[1] Aerodonetics, London, 1908.

results will be evident on examining the discussion of the long oscillations and their trajectories given in §§ 52, 53 of this book. In applying approximate methods to separate the long oscillations, two successive approximations are shown to be necessary. The first approximation determines the period of oscillation, and the trajectory of the aeroplane is shown to this order to be of the character described by Lanchester, and to be unaffected by the rotatory inertia of the mass. To determine the modulus of decay a second approximation taking account of the rotatory inertia is necessary, and the fact that the modulus of decay is of a higher order than the frequency justifies the averaging method adopted by Lanchester. But as a necessary consequence, of course, Lanchester's method does not give the short oscillations or even indicate their existence. The sufficiency of his condition depends on the fact, shown in § 51, that the conditions of stability for the long oscillations necessarily involve the conditions of stability for the short ones.

In the case of "Lateral and Directional Stability," Lanchester does not use the three equations of motion, but he is careful to emphasize the interdependence of the three motions which would be determined by these equations. He also emphasizes the necessity of fins or bent-up planes, or both if an aeroplane is to be stable. Whether the stability is to be described as "rotative" or "asymmetric" is a question of terminology. Under the heading of "Fin Resolution" Lanchester gives a discussion practically amounting to the same thing as our application of the principle of Parallel Axes to the second moments of the fins. This method shows the essential difference between a single fin and a pair of fins, one in front and one behind, in the matter of producing stability. The coefficients which we denote by I are expressed in terms of two quantities called the aerodynamic and aerodromic radii. (See § 93 above for a

discussion of the effect of assuming them unequal.) The stability in non-horizontal flight is not discussed, but an experiment is described in which a model turned suddenly round sideways at particular points of its flight-path, and the conclusion is suggested that owing to the longitudinal oscillations the straight path became laterally unstable at those points where the inclination of the flight path exceeded a certain limit for stability.

BRILLOUIN.[1]

108. Prof. Marcel Brillouin approaches the question of balance and stability from a different point of view. His investigations refer mainly to *static* instead of dynamic instability, and he seeks by the study of metacentric curves to discuss the conditions that an aeroplane shall have a unique position of statically stable equilibrium, and thus be self-righting in all positions. He proceeds to study the effect of manœuvring on the balance of an aeroplane, and his investigation thus becomes independent of the problem of *inherent dynamical stability*.

REISSNER.[2]

109. In the first of two recent articles of considerable interest, Dr. H. Reissner, of Aachen, places the problem of *lateral steering* of aeroplanes on an exact and formal basis. In a subsequent article, he points out the insufficiency of the conditions for lateral stability previously proposed, and the necessity of a further condition, thus confirming the conclusions of the present investigation.

CROCCO.[3]

110. Lieutenant Crocco's principal paper refers to dirigibles, not aeroplanes. In investigating the longi-

[1] *Stabilité des Aeroplanes, surface métacentrique. Revue de Mécanique*, 1909.

[2] *Zeitschrift für Flugtechnik und Motorluftschiffahrt*, 1910, 9, 10, and *Flugsport*, 1910–11.

[3] *Bollettino della Societa aeronautica italiana*, April, 1907, and June, 1907. *Government Blue Book*, 1909, 10, pp. 41, 161.

tudinal stability, he obtains a cubic instead of a biquadratic, based on the equations of rotation and of vertical motion. The equation of horizontal motion is thus ignored. This is certainly incorrect in the case of an aeroplane. Both Lanchester's and the present investigation show that in the long longitudinal oscillations the variations in horizontal velocity play a predominating part, and, moreover, we have proved that the conditions of stability depend more on the long than on the short oscillations, stability for the former in general sufficing for the latter. For directional stability, Crocco again obtains a cubic, which is certainly inapplicable to aeroplanes. Even in the case of a dirigible, Crocco's fundamental assumption, "that the velocity is constant within the time considered," is open to serious question. In another note, attention is called to the stability condition which requires that for a single lifting aeroplane the neutral plane must be behind. It is pointed out that for a double lifting aeroplane the condition $a_1 - a_2$ positive has an equivalent effect.

Soreau.[1]

111. For longitudinal stability, Soreau certainly obtains a biquadratic, and deduces conditions analogous to those obtained in the present investigation. In connection with asymmetric stability, he falls into the common error of not recognizing the interdependence of stability for rolling and directional stability, thus applying the term lateral stability in its most restricted sense. He is thus led to conclusions with which the present writer altogether disagrees, and which are further in conflict with the trend of Lanchester's chapter on the subject. Soreau would do away with vertical auxiliary surfaces and bent-up planes, and would, in particular, avoid vertical

[1] *État actuel et l'Avenir de l'Aviation.* (*Memoires de la Société des Ingenieurs civiles de France*, 1908, II. *Blue Book*, 1909-10, pp. 54, 154.)

auxiliary surfaces raised above the centre of gravity (see Blue Book, p. 66). In other words, he would leave his machine *inherently unstable*, and would use a gyroscope or other similar contrivance to counteract automatically this defect.

LECORNU.[1]

112. The Government abstracts [2] contain a reference to Lecornu's paper on "Graphic Statics," which, like our discussion in §§ 36–47, deals with the application of graphic methods to the forces acting on an aeroplane, the sine law of resistance being assumed.

[1] *Sur la statique graphique de l'aeroplane.* (*Comptes rendus*, February 22, 1909.)

[2] *Blue Book*, 1909–10.

CHAPTER XI.

PROBLEMS.

113. The following problems refer partly to questions not coming within the scope of this book, and partly to points which it has been considered scarcely desirable to discuss in further detail in the preceding pages. Some are pretty simple, and could be treated as exercises or research subjects for a science student of fair ability; others would involve mathematical investigations or experiments, or both, of a prolonged character.

1. The "characteristics" of a system of ideal planes following the sine law of resistance, *i.e.* the determination of the simplest possible scheme of equivalent planes (see note on Captain Ferber, § 106)—a purely mathematical investigation.

2. The steering of an aeroplane in a horizontal circle. A study of Dr. Reissner's paper on this subject (§ 109) will certainly show possibilities of further investigation.

3. The stability of a dirigible, and in this connection the discussion of formulæ that shall best represent the resistances due to inertia combined with head resistance, in view of the fact that the hydrodynamical formulæ for inertia effects refer, of course, to motion in an incompressible, unresisting medium. The six equations of motion to be used, and not to be assumed independent unless shown to be so.

4. A determination of the initial motions set up when an aeroplane is suddenly struck by a gust of wind in any

direction, longitudinal, lateral, or vertical, or compounded of all three.

5. The calculation of the stresses set up in the framework of the aeroplane in such a case.

6. An experimental investigation on the effects of rotation on broad planes, and, if possible, the determination of formulæ, empirical or otherwise, which shall represent these effects at least approximately.

7. Experiments on the wash caused by front planes on back planes, also on the effects of wash caused by propellers.

8. The effect of a low centre of gravity on longitudinal stability, *i.e.* a further development of the methods suggested in § 69, for an aeroplane the planes and propeller of which are both raised to the same height.

9. An experimental study of the effects of inclination of flight path on stability.

10. A more detailed examination of the effects of the "product of inertia" F on lateral stability. It will be seen that in our investigation F has been retained in most of the fundamental formulæ, but that in the final deductions we have usually put $F = 0$, thus assuming the line of flight to be a principal axis of inertia. As a deviation from this condition may either increase or decrease the stability, the investigation may become important.

11. The lateral stability of types like the Langley aerodrome, consisting of two pairs of bent-up wings arranged tandem. The mathematics is in this case bound to be fairly heavy.

12. A more extended descriptive examination of the nature of the asymmetric oscillations and the motions associated with them.

13. The investigation of the forced oscillations set up in an aeroplane by periodic gusts of wind, with especial reference to the case of synchrony with the free oscillations and the influence of damping in this case.

14. The possibility, or otherwise, of helicoidal steady motion in the case of a symmetrical aeroplane, in particular when steady motion in a straight line is laterally unstable.

15. A further discussion of the effects of camber on both longitudinal and lateral stability.

16. A complete mathematical investigation (in the form of a thesis) on the stability and oscillations of a kite, with special reference to the question as to how far the resistance derivatives and stability of an aeroplane could be investigated experimentally by flying the model as a kite.

17. Experimental determinations of the resistance derivatives of aeroplane models by attaching them to whirling tables or pendulums, with especial reference to those derivatives which depend on rotation, comparison with calculated values, experimental flights with the same models, and comparison of their behaviour as regards stability with the results of calculation.

18. Further photographic determinations of the paths and oscillations of model aeroplanes and gliders, with special reference to their lateral oscillations.

19. The graphic statics of longitudinal equilibrium for systems of *three* or more planes having different abscissæ; effect of altering the inclination of one of them relative to the aeroplane.

20. Finally, the discussion of the equations of motion of an aeroplane in their most general form, and the search for cases in which these equations admit of exact integration, is probably a mathematical problem which, like the well-known Problem of Three Bodies, may well occupy the attention of pure mathematicians for the next half century, and still leave plenty of work to be done.

Examples.

114. Particulars of the construction and dimensions of the leading modern aeroplanes will be found in most aeronautical journals and manuals and engineering papers. An abundant supply of "examples" can be obtained by calculating their stability by the methods of this book; determining the coefficients of the biquadratics for longitudinal and lateral oscillations; approximate solutions of the biquadratics; periods and logarithmic decrements (or, in the case of instability, logarithmic increments) of the principal oscillations.

NOTES.

Gyrostatic action of propellers.

115. The gyrostatic effects due to the angular momentum of a single propeller not only affect the steering of the aeroplane, as is well known, but introduce interdependence between the equations of rotation about the axes of y and z, thus mixing up the longitudinal and lateral oscillations. In a rigorous investigation, all six equations of motion would have to be considered simultaneously.

The best plan is undoubtedly to destroy the gyrostatic couples either by the use of twin screws rotating in opposite directions, or by following the suggestion of a recent French inventor who makes the rotary engine and the propeller rotate in opposite directions, so that their gyrostatic effects cancel. A pair of cog wheels is all that is required.

If the gyrostatic couple is small, a solution of the problem could be obtained, if it were worth while to do so, by approximate methods. For a first approximation, we should neglect the effect altogether, so that the oscillations would fall into the two "symmetric" and "asym-

metric" groups. We should then apply small corrections for the asymmetric motions set up by the symmetric oscillations and *vice versâ*. If an asymmetric and a symmetric oscillation happened to have nearly the same period, "resonance," effects would be set up with which physicists are familiar, and the aeroplane would probably not be a very comfortable one to travel on.

A proposition on automatic control.

116. The name "automatic stability" is now applied to devices for regulating the balance of aeroplanes by means of gyrostats or other mechanical contrivances. In this connection, the following proposition may be useful :—

An aeroplane formed of ideal straight planes with no vertical fins or bent-up surfaces cannot be rendered laterally stable by the use of gyrostats or pendulums alone.

This applies even to the case where the aeroplane is provided with warping devices or ailerons, if these are only brought into action by the operations of the gyrostat.

Proof.—In order that the plane of symmetry should tend to assume a vertical direction when the machine is disturbed, the gyrostat must be sensitive to variations in the direction of gravity relative to the machine, since the vertical direction is entirely defined by gravity.

Now, suppose such a machine (initially moving horizontally) to be heeled over sideways through an angle ϕ. By assumption, there are no tangential resistances, therefore the machine will slip sideways with acceleration $g \sin \phi$. The only effect which the gyrostat is able to detect is equivalent to a change in the intensity of gravity from g to $g \cos \phi$, the direction of the latter component *relative to the machine* being unaltered. There is nothing, therefore, to make the gyrostat bring the machine back to the vertical.

The only remedy is to retard the acceleration $g \sin \phi$, or make it evident in some way or other. This can only be done by the use of some kind of fins or stabilizers or other auxiliary surfaces attached to the machine or to the gyrostat. In any case, it is impossible to do this without in an equal degree making the gyrostat sensitive to changes in wind velocity. When this is admitted, we might just as well do away with the gyrostats altogether and rely on the auxiliary surfaces instead.

One further point should, however, be considered. An aeroplane might be provided with auxiliary surfaces calculated to make it *more inherently unstable*, and gyrostats might be used to convert this instability into stability, as is, indeed, actually done in the Brennan monorail. What is here implied is that if an aeroplane, when tilted over sideways, tends either to return to the vertical position or to deviate further from it, gyrostats may be used with effect, but if it possesses neither tendency, they become ineffectual, and some kind of auxiliary surfaces must necessarily be introduced.

Note on the critical inclinations.

117. In discussing the effects of inclination of flight path on stability we came upon the two limiting relations $\tan \theta + 2 \tan \alpha = 0$ for longitudinal, and $\tan \theta - 2 \tan \alpha = 0$ for lateral stability. It is useful therefore to get *some rough idea* as to the why and wherefore of such condition, and their dependence on the law of resistance $R = KSU^2 \sin \alpha$.

Consider a single plane area the pressure on which obeys this law. Draw the curve $r^2 \sin \theta = \text{constant}$. Then as long as the relative wind velocity is represented by a radius vector of this curve, the pressure remains constant, and if the plane receives a small additional velocity dv in a direction *tangential* to the curve the same

thing occurs. If the direction of dv falls within the curve the pressure decreases, if without, it increases. The angle ϕ (Fig. 38) between U and the direction of dv, when the latter is tangential, is $\tan^{-1}(-r d\theta/dr) = \tan^{-1}(2 \tan \theta)$.

If now the area is a kite placed in a horizontal wind, α being the inclination of its face to the horizon, and if the kite is kept in place by a string whose inclination to the vertical is less than the angle whose tangent $= 2 \tan \alpha$, it will be seen that a small additional upward velocity perpendicular to the string will increase the pressure and cause the kite to rise further, while a small velocity

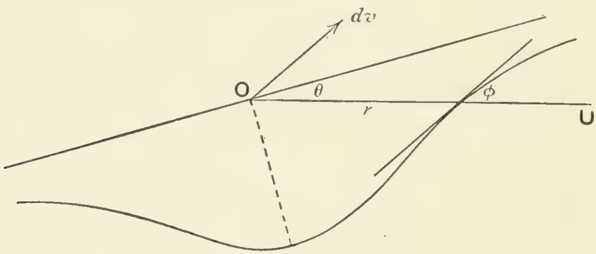

Fig. 38.

in the opposite sense will reduce the pressure and cause the kite to fall. This result indicates instability. For stability the inclination of the string to the vertical must be greater than $\tan^{-1}(2 \tan \alpha)$. A complete discussion of the conditions of stability would, however, necessitate taking account of equations of rotation as well as translation.

We may similarly show, with the assumptions of § 76, that if the main plane of an aeroplane receives a small rotation in a plane whose inclination to the line of flight is given by $\tan^{-1}(2 \tan \alpha)$ no couples are set up. The limiting condition $\tan \theta = 2 \tan \alpha$ for lateral stability thus represents the fact that no couples are set up if the aeroplane receives a small rotation in a horizontal plane.

The same thing can be seen by means of the general expressions (23) of § 20. The condition $\mathfrak{E}_1 = 0$ may be written

$$\begin{vmatrix} 0, & \cos\theta, & -\sin\theta \\ L_w & L_p & L_q \\ M_w & M_p & M_q \end{vmatrix} = 0,$$

and is the condition that the equations

$$\begin{aligned} p\cos\theta - q\sin\theta &= 0 \\ wL_w + pL_p + qL_q &= 0 \\ wM_w + pM_p + qM_q &= 0 \end{aligned}$$

should be compatible; in other words, that the motion characterised by w, p, q which gives rise to no couples L, M, should be a rotation about a vertical axis, as indicated by the condition $p \cos\theta - q \sin\theta = 0$.

The limiting condition, $\tan\theta = -2\tan\alpha$, for longitudinal stability arising out of the expression for $\mathfrak{D}_o$, is more difficult to explain. If we revert to the general equations (18) assuming $\Delta_o = 0$ and $X_u = 0$ the condition becomes $W \sin\theta + UX_u = 0$ or $W \sin\theta + 2X_o = 0$, and leads to the statement that for stability the resistance due to the component of gravity in the line of flight must be less than twice the resistance due to the air. The effect of head resistance is to increase the latter, and this is why it increases the angle at which the aeroplane can rise.

NOMENCLATURE.

118. **Abscissa.**—In accordance with our system of coordinates, the *abscissa* of any plane is its distance in front of the centre of gravity, being negative if the plane is behind.

Aeroplane and plane.—In accordance with the Aeronautical Society Committee's recommendation, aeroplane = a flying machine, while plane = one of its surfaces. In most cases in this book a pair of superposed planes counts as a single plane for reasons explained.

Angle of attack.—This is the angle at which the air blows on any given surface; in an aeroplane moving steadily it is the angle which a plane makes with the direction of flight. It is sometimes called the angle of incidence, but the optical analogy suggests that this name would better be applied to the angle which the air current makes with the *normal* to the surface.

Axes of co-ordinates.—The centre of gravity G is taken as origin. The line of flight in steady motion is the axis of x; a perpendicular line in the plane of symmetry measured *downwards* is the axis of y, and a perpendicular to the plane of symmetry is the axis of z, the three forming a right-handed system.

Fin.—A vertical auxiliary plane.

Inherent and automatic stability.—The term *automatic stability* is now frequently applied to mechanical devices with movable parts, such as gyrostats, which are so arranged as to restore equilibrium when an aeroplane or other system is displaced. The Brennan monorail affords one of the simplest illustrations of automatic stability. An aeroplane is *inherently* stable when stability is secured without the aid of movable parts, and the only kind of stability considered in this book is *inherent dynamical stability.*

Narrow planes.—Planes of which the chord or breadth is so small that when the plane rotates about an axis parallel to the axis of z the difference of velocity of the front and back edges is negligible, *i.e.* the rotary derivatives are negligible.

Plane of symmetry.—Aeroplanes, practically without exception, are symmetrical. When flying in a straight line the plane of symmetry is a vertical plane through the centre of gravity containing the line of flight.

Resistance derivatives.—If small changes be made in the velocity components of an aeroplane, the ratios to these of the corresponding changes in the values of the components of air resistance I call the *resistance derivatives.*

Rotary derivatives.—The resistance derivatives arising from a small rotation of a plane about an axis through its centre of pressure parallel to the axis of z.

Single and double lifting systems. Neutral surfaces.—Where an aeroplane has two surfaces or sets of superposed surfaces, and the weight of the aeroplane is borne partly by the front and partly by the rear surfaces, I call it a *double lifting system.* If the whole weight is borne by a single surface or set of superposed surfaces, I call it a *single lifting system.* In such cases any other auxiliary surfaces, such as tail planes, must be parallel to the direction in which the wind blows on them, and I call this position *neutral.*

Small angles.—When the angle of attack is called *small*, it is implied that the resistance may be taken to follow the "sine law."

Stability, static and dynamic.—A system in equilibrium is said to be *statically stable* if, when displaced, it initially tends to approach its original state of equilibrium. It may, however, go past its equilibrium position and oscillate, and if the oscillations continually grow larger and larger it will be *dynamically unstable.* For *dynamical stability* it is necessary that the system should either return to its equilibrium position without oscillating, or that the oscillations should gradually die out. In the present book "stability" means dynamical stability.

Stabilizer.—In this book the term is applied to small bent up planes attached at the extremities of the main planes.

Straight and bent up planes.—When the wings of an aeroplane form a dihedral angle, I call them "bent up." "Straight" planes, on the other hand, project perpendicularly on both sides of the plane of symmetry.

Symmetric and asymmetric stability. — *Symmetric stability* is practically synonymous with longitudinal stability, and represents stability for motions in the plane of symmetry. *Asymmetric stability* is stability for rotations about the axis of x and y and displacements along the axis of z, and includes what are sometimes separated as "lateral" and "directional" stability. It is Lanchester's "rotative stability."

119. NOTATION.

A, B, C, moments of inertia of aeroplane. F, product of inertia.

$\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$, co-efficients of biquadratic, with suffix $_0$ for symmetrical, $_1$ for asymmetric oscillations where a distinction is necessary.

$\boldsymbol{a}$, angle of attack.

$\boldsymbol{a}_1$, $\boldsymbol{a}_2$, angles of attack in a double lifting system, and in this case $\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2$.

$\boldsymbol{\beta}$, angle at which a plane is bent up.

a, semi-chord of plane.

b, semi-span.

c, radius of curvature of cambered plane.

d, distance of centroid of areas of a pair of transverse planes from centre of gravity.

f, corresponding distance of centre of pressure of the pair.

e, a constant determining the displacements of the centre of pressure of a transverse plane.

ϵ, a certain constant referring to the "Antoinette" type (under lateral stability).

F, product of inertia of aeroplane as above.

$f(\alpha)$, function of angle of attack determining law of resistance; $\phi(\alpha)$, corresponding function determining position of centre of pressure.

$f'(\alpha)$, $\phi'(\alpha)$, differential coefficients with respect to α; $f_r(\alpha)$, $\phi_r(\alpha)$, rotary derivatives corresponding to a small angular velocity r.

H, propeller thrust; h, its perpendicular distance from origin; η, its inclination to axis of x when different from zero.

I, moment of inertia of a transverse plane with respect to plane of symmetry $I' = I \cos^2 \alpha$.

K, coefficient of resistance in $R = KSU^2 f(\alpha)$, or $R = KSU^2 \sin \alpha$.

k, radius of gyration about axis of z, $C = Wk^2$.

l, distance between front and rear planes, in longitudinal stability.

l, m, n, direction cosines of normal to a bent up surface-element.

L, M, N, couples about axes due to resistance, in directions opposed to rotation.

L_o, M_o, N_o, their values in steady motion.

L, M, N, with suffixes, *e.g.* L_p the corresponding resistance derivatives; $L_p = dL/dp$.

M_1, M_2, P moments and products of inertia of a system of two or more vertical fins with respect to axes through their centre of pressure parallel to the co-ordinate axes, the co-ordinates of this centre of pressure being x, y, z.

λ, coefficient in a small oscillation or disturbance proportional to $e^{\lambda t}$, hence a root of the biquadratic.

μ, tangent of angle of attack $\mu = \tan \alpha$.

μ_1, μ_2, tangents of angles of attack in double lifting systems.

ν, the difference of these tangents $= \mu_1 - \mu_2$.

p, q, r, components of angular velocity of aeroplane.

R, resultant thrust of air on a plane.

R, radius of gyration of a pair of planes (where this notation causes no confusion with the previous one).

ρ distance measured from dihedral angle on a bent up plane.

S, area of any plane surface, $S' = S \cos^2 a$.

T, area of a vertical fin or pair of stabilizers, also total area of several fins.

U, velocity in steady flight.

u, v, w, added velocity components in a small oscillation.

W, weight of aeroplane.

x, y, z, co-ordinates.

X, Y, Z, components of force due to air resistance, taken positive when acting in the opposite direction to the axes. X_o, Y_o, Z_o, values in steady motion. X_u, etc., corresponding resistance derivatives $X_u = dX/du$.

ζ, a certain constant in connection with stabilizers.

θ, inclination of flight path to horizon, positive when descending; in small symmetric oscillations $\theta = \theta_o + \epsilon$ where θ_o is the value corresponding to steady motion. This notation causes no confusion with the other meaning of ϵ mentioned above.

ν, ratio of head resistance to drift.

ϕ, angle through which plane of symmetry has rotated about axis of x, measured from vertical position.

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